1. Let $(M, d)$ be a metric space. Show the following inequality:

$$|d(x, y) - d(z, y)| \leq d(x, z), \quad \forall x, y, z \in M.$$ 

2. Let $A$ be an open set in $\mathbb{R}$, and define the set $B := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in A\} \subseteq \mathbb{R}^2$. Show that $B$ is open in $\mathbb{R}^2$.

3. Let $(M, d)$ be a metric space. Given a nonempty set $A \subseteq M$, let $B := \{x \in M \mid d(x, y) < 1 \text{ for some } y \in A\}$. Show that $B$ is open. (Hint: write $B$ as the union of open sets.)

4. Consider $\mathbb{R}^2$ and the metric induced by the 1-norm: $d(x, y) = |x_1 - y_1| + |x_2 - y_2|, \forall x, y \in \mathbb{R}^2$. Let $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 1 \text{ and } x_2 \leq 1\}$. Find the interior of $A$ (using the given metric), and prove your answer.

5. Let $(M, d)$ be a metric space and $A, B$ be two subsets of $M$. Show the following:

   (1) if $A \subseteq B$, then $\text{int}A \subseteq \text{int}B$;
   (2) $\text{int}(A \cap B) = (\text{int}A) \cap (\text{int}B)$.

The following extra problems are for Math 600 students only:

6. Given a normed space $(V, \| \cdot \|)$, let $A$ be a nonempty open set in $V$, and $B$ be a nonempty set in $V$. Define $A + B := \{a + b \in V \mid a \in A, b \in B\}$. Show that $A + B$ is open.

7. Let $M$ be a set endowed with two metrics $d_1$ and $d_2$, namely, both $(M, d_1)$ and $(M, d_2)$ are metric spaces. Suppose that there exist two positive real numbers $\alpha$ and $\beta$ such that

$$\beta d_1(x, y) \leq d_2(x, y) \leq \alpha d_1(x, y), \quad \forall x, y \in M.$$ 

Show that a set $A \subseteq M$ is open with respect to $d_1$ if and only if $A$ is open with respect to $d_2$. 

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\| \cdot \|_2$ on $\mathbb{R}^n$, unless otherwise stated.