Math 600  Fall 2015  Homework #7

Due Dec. 1, Tue.  in class

Note: For the Euclidean space \( \mathbb{R}^n \), consider the usual metric induced by the Euclidean norm \( \| \cdot \|_2 \) on \( \mathbb{R}^n \), unless otherwise stated.

1. Let \( f_n : [1, 2] \to \mathbb{R} \) be \( f_n(x) = \frac{x}{(x+1)^n} \).
   (1) Use the Weierstrass M-test to show that \( \sum_{n=1}^{\infty} f_n(x) \) is uniformly convergent on \( A = [1, 2] \);
   (2) Determine if \( f_1(\sum_{n=1}^{\infty} f_n(x))dx = \sum_{n=1}^{\infty} f_1 f_n(x)dx \).

2. Let \( A = [-a, a] \subset \mathbb{R} \) with \( a > 0 \), and let
   \[ f_n(x) = \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}, \quad x \in \mathbb{R}. \]
   (1) Use the Weierstrass M-test to show uniform convergence of the series \( \sum_{n=1}^{\infty} f_n \) on \( A \);
   (2) Let \( f_* \) be the limit function of the series on \( A \), i.e., \( f_* (x) = \sum_{n=1}^{\infty} f_n(x) \). Is \( f_* \) differentiable on \(( -a, a ) \)? If so, is \( f'_*(x) = \sum_{n=1}^{\infty} f'_n(x) \) on \(( -a, a ) \)? Prove your answers.

3. Let \( f_n : \mathbb{R} \to \mathbb{R} \) be
   \[ f_n(x) = \frac{(-1)^{n+1}x}{n}. \]
   Let \( A \) be a bounded set in \( \mathbb{R} \). Show that the series \( \sum_{n=1}^{\infty} f_n(x) \) converges uniformly on \( A \).
   (Hint: use the Cauchy criterion.)
   * Note that the Weierstrass M-test fails for this series. This example shows that the Weierstrass M-test is a sufficient condition for uniform convergence but not a necessary one.

4. Use the Cauchy criterion to show that the series \( \sum_{n=1}^{\infty} x^n \) does no converge uniformly on the open interval \(( -1, 1 ) \). (Hint: let \( s_n \) be the \( n \)th partial sum, and for any \( m > n \), look at \( |s_m(1) - s_n(1)| \) and approximate 1 by a suitable \( x \in ( -1, 1 ) \).)

5. Let \( f_n : \mathbb{R} \to \mathbb{R} \) be \( f_n(x) = \frac{x}{n^2 + x^2} \).
   (1) Use the Weierstrass M-test to show that the series \( s_* := \sum_{n=1}^{\infty} f_n \) converges uniformly on any bounded set \( A \subset \mathbb{R} \). Furthermore, show that \( s_* \) is continuous at any point in \( \mathbb{R} \).
   (2) Show that the series \( \sum_{n=1}^{\infty} f_n \) does not converge uniformly on \( \mathbb{R} \) via the Cauchy criterion.

6. Let the constant \( K \) satisfy \( 0 < K < 1 \). Consider the linear function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by
   \[ f(x) = \frac{K}{\sqrt{2}} (x_1 + x_2, \ x_2 - x_1), \quad \forall x = (x_1, x_2) \in \mathbb{R}^2. \]
   In the following, you may use the results of Problem 4.
   (1) Show that when the 2-norm (i.e., \( \| \cdot \|_2 \)) is used, \( f \) is a contraction.
   (2) Show that when the 1-norm (i.e., \( \| \cdot \|_1 \)) is used, \( f \) is not a contraction if \( \frac{1}{\sqrt{2}} < K < 1 \).
   (3) Let \( x^0 = (x^0_1, x^0_2) \in \mathbb{R}^2 \) be arbitrary. Define the sequence \( (x^k) \) as \( x^k = f(x^{k-1}) \), \( k \in \mathbb{N} \). Explain why the sequence \( (x^k) \) is convergent when the 2-norm is used. (Note: recall that \( (\mathbb{R}^2, \| \cdot \|_2) \) is complete.)
Show that the sequence defined in (3) is convergent when the 1-norm is used. (Hint: use the equivalence of norms on a Euclidean space shown in Homework #5.)

⋆ This example shows that the contractive property is a sufficient condition for convergence but not a necessary one.

Miscellaneous practice problems: Do not submit

1. Let \( f_n : \mathbb{R} \to \mathbb{R} \) be such that the sequence \((f_n)\) converges uniformly to \( f_* \) on the set \( A \). Suppose that each \( f_n \) is bounded on \( A \), i.e., for each \( f_n \), there exists \( M_n > 0 \) (dependent on \( f_n \)) such that \( |f_n(x)| \leq M_n, \forall x \in A \). Show that \( f_* \) is bounded on \( A \).

2. Find the largest possible constant \( r \in (0, 1) \) such that the function \( f : [0, r] \to [0, r] \) defined by \( f(x) = x^2 \) is a contraction.

3. Let \((V, \| \cdot \|)\) be a complete normed vector space and its induced metric \( d(x, y) = \|x - y\| \) for \( x, y \in V \). Let \( f : V \to V \) be a linear mapping/function, i.e., \( f(x + y) = f(x) + f(y), \forall x, y \in V \) and \( f(\alpha x) = \alpha f(x) \) for all \( x \in V \) and \( \alpha \in \mathbb{R} \). You may assume the following facts without proof: \( f(0) = 0 \) and \( f(x - y) = f(x) - f(y), \forall x, y \in V \).

   (1) Show that \( f \) is a contraction if and only if there exists a constant \( C \) with \( 0 < C < 1 \) such that \( \|f(x)\| \leq C\|x\| \) for all \( x \in V \).

   (2) Suppose that \( f \) is a contraction. Let \( x_0 \in V \) be arbitrary, and define the sequence \((x_n)\) recursively by \( x_n = f(x_{n-1}), \ n \in \mathbb{N} \). Show that \((x_n)\) converges to the zero vector in \( V \).