This is a 120 min exam.

Problem 1: (Solution Sets) 15 POINTS  Show all Work

\[
\begin{align*}
    x_1 + 2x_2 - x_3 &= 4 \\
    x_1 + 3x_2 + x_3 &= 1 \\
    3x_1 + 7x_2 - x_3 &= c
\end{align*}
\]

For what values of c does the system have
   a) No solutions
   b) One unique solution
   c) Infinitely many solutions
Problem 2: (Solving Linear Systems) 15 POINTS  Show all Work

\[ \begin{align*}
  x_1 -3x_2 -9x_3 &= 0 \\
  x_2 +2x_3 &= 0
\end{align*} \]

(a) Find the general solution of the above homogeneous system of equations. Express your answer in parametric vector form.
(b) What is the dimension of the null space and the column space of the matrix \( A \)? Explain!
Problem 3 (LU Factorization): 15 POINTS  Show all Work

\[
A = \begin{bmatrix}
1 & 3 & 4 \\
-1 & -8 & -9 \\
2 & 1 & 2 \\
\end{bmatrix}
\]

(a) Find an LU factorization for the following matrix:
(b) What does \( U \) tell us about the invertibility of \( A \)?
Problem 4 (Orthogonality) : 15 POINTS Show all Work

Let \( u = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} \), and \( v = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \).

1. Compute \( uv^T \).

2. Find the distance between \( u \) and \( v \).

3. Find a unit vector in the direction of \( v \).

4. Find a vector orthonormal to the unit vector in the direction of \( v \).

5. Do \( \{u, v\} \) form a basis for \( \mathbb{R}^3 \)? \( \mathbb{R}^2 \)? Explain!!
Problem 5 (Diagonalization): 15 POINTS  Show all Work

Find $P, D, P^{-1}$ by diagonalizing the following matrix: $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
Problem 6 (Vector Spaces): 15 POINTS  Show all Work

Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace $U$ of $V$. Let $T(U)$ denote the set of all images of the form $T(x)$, where $x \in U$.

(a) What 3 properties does as subspace $U$ of $V$ have?

(b) What 3 properties should $T(U)$ have to be a subspace of $W$?

(c) What is the definition of a linear transformation?

(d) Draw a picture representing $U \subseteq V$, $T(U) \subseteq W$ and $T : V \to W$.

(e) Show that $T(U)$ is a vector subspace of $W$.
   (hint: Verify $T(U)$ has all the properties listed in part (b) using part (a) and part (c))
Problem 7 (Change of Basis): 15 POINTS  Show all Work

Let

\[
\begin{align*}
    b_1 &= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \\
    b_2 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
    b_3 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
    x &= \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}
\end{align*}
\]

(a) Show that \( B = \{b_1, b_2, b_3\} \) is a basis of \( \mathbb{R}^3 \).
(b) Find the change of coordinates matrix from \( B \) to the standard matrix.
(c) Write the equation that relates \( x \) in \( \mathbb{R}^3 \) to \([x]_B\)
(d) Find \([x]_B\) for the \( x \) given above.
Problem 8 (Determinants): 15 POINTS  Show all Work

\[ A = \begin{bmatrix} 3 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \\ -1 & 3 & 0 & 0 \end{bmatrix} \]

(a) Calculate the determinant of \( A \) using row operations and determinant properties.
(b) Calculate the determinant of \( A \) using cofactors expansion to check part (a)
(c) Do the columns of \( A \) span \( \mathbb{R}^4 \)? (Why or Why not)

\[ B = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \]

(d) Is this matrix invertible? If yes, for which \( \theta \)? and calculate inverse, If no, Why.
(Extra credit +5: Is \( B \) orthonormal If yes, show it, If no, Why?)
Problem 9 (True of False): 15 POINTS Show all Work
True/False (if True, why. If False, give counter or explain)

(a) The equation $Ax = 0$ has the trivial solution only if there are not free variable.

(b) If $A$ is a $m \times n$ matrix with $m$ pivot columns, then the linear transformation $x \mapsto Ax$ is a one to one mapping.

(c) If $A$ and $B$ are $n \times n$ then $(A + B)(A - B) = A^2 - B^2$

(d) If $AB = BA$ and if $A$ is invertible, then $A^{-1}B = BA^{-1}$

(e) Suppose $A$ is $n \times n$, then $det(-A) = -detA$.

(f) Suppose $A$ and $B$ are $n \times n$ matrices, if $AB = 0$ then $det(A) = 0$ or $det(B) = 0$.

(h) If $B$ is an echelon form of a matrix $A$, then the pivot columns of $B$ form a basis for the $ColA$. 

Problem 10 (Invertibility): 15 POINTS  Show all Work

(a) List eight equivalent statements stated in the Inverse Matrix Theorem.

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \]

(b) Invert \( A \) using Gaussian Elimination.
(c) Invert \( A \) using Cramer’s Rule.
Problem 11 (Linear Transformations): 15 POINTS  Show all Work

Let $T : \mathbb{P}_2 \rightarrow \mathbb{P}_3$ be the transformation that maps a polynomial $p(t)$ into a polynomial $tp(t)$.

(a) Find the image of $p(t) = 2 - t + t^2$ under this transformation.
(b) Show that $T$ is a linear transformation.
(c) Find the matrix for $T$ relative to the basis $\{1, t, t^2\}$.
(d) Is $T$ one to one?
(e) Is $T$ onto?
(f) Is $T$ invertible? (explain)