Problem 1: (10 min) Euler’s Method 10 POINTS  Show all Work!

\[
\frac{dy}{dx} = y \quad y(0) = 1
\]

a) Use Euler’s Method to approximate the solution of this IVP (namely \(e^x\)) at the point \(x = 1\). Assume \(h = 0.25\)

\[
y_{n+1} = y_n + hy_n \\
y_{n+1} = y_n(1 + h) \\
y_4 = y_0(1 + h)^4 \\
y_4 = (1.25)^4 = \frac{625}{256}
\]

b) \(e^1 = 2.718281828\). So what is \(\frac{\text{error}}{h}\) ?

Note: The error is the (True value - Euler’s approximation).

\[
(2.718281828 - \frac{625}{256}) \times 4
\]

c) What order is Euler’s method? and hence what do you expect to happen to the ratio \(\frac{\text{error}}{h}\) as \(h \rightarrow 0\)?

Order 1: \(\frac{\text{error}}{h} \rightarrow \text{constant as } h \rightarrow 0\)
Problem 1: Euler’s Method Workspace
Problem 2: (20 min) Auxiliary equations and Implication of Roots
25 POINTS  Show all Work!

\[ y'' + 2\alpha y' + y = e^{-t} \quad y(0) = 1 \]

a) Write down the auxiliary equation for the corresponding homogeneous system of the IVP above.

\[ r^2 + 2\alpha r + 1 = 0 \]

b) Determine the values of \( \alpha \) so the auxiliary equation has:

\[ \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4}}{2} = -\alpha \pm \sqrt{\alpha^2 - 1} = r_{1,2} \]

Case 1: A double real root

\[ \alpha = \pm 1 \Rightarrow r = -\alpha \Rightarrow r = \mp 1 \]

Case 2: Two real roots

\[ \alpha^2 - 1 > 0 \Rightarrow \alpha^2 > 1 \Rightarrow |\alpha| > 1 \]

Case 3: Complex conjugate pair roots

\[ \alpha^2 - 1 < 0 \Rightarrow \alpha^2 < 1 \Rightarrow |\alpha| < 1 \]

c) What are the roots for each case above.

Please leave \( \alpha \) as a parameter in your answers but describe its constraint in each case.

Case 1: \( r = 1 \) for \( \alpha = -1 \), \( r = -1 \) for \( \alpha = 1 \)

Case 2: \( r_{1,2} = -\alpha \pm \sqrt{\alpha^2 - 1} \) for \( |\alpha| > 1 \)

Case 3: \( r_{1,2} = -\alpha \pm i\sqrt{1 - \alpha^2} \) for \( |\alpha| < 1 \)

d) Write down the homogeneous solution for each case above.

Case 1: \( y_h = c_1 e^{-t} + C_2 te^{-t} \) and \( y_h = c_1 e^{1t} + C_2 e^{1t} \)

Case 2: \( c_1 e^{-\alpha + \sqrt{\alpha^2 - 1}} + c_2 e^{-\alpha - \sqrt{\alpha^2 - 1}} \)

Case 3: \( e^{-\alpha t}(c_1 \sin(\sqrt{1 - \alpha^2} t) + c_2 \cos(\sqrt{1 - \alpha^2} t)) \)
Problem 2: Auxiliary equations and Implication of Roots Workspace
Problem 3: (20 min) Method of Undetermined Coefficients 25 POINTS  Show all Work!

\[ y'' + 2\alpha y' + y = e^{-t} \quad y(0) = 1 \]

a) Determine the particular solution to the ODE above if

\( \alpha = 0 \): plug guess into ODE

\[ y_p = Ae^{-t} \Rightarrow Ae^{-t} + Ae^{-t} = e^{-t} \Rightarrow A = \frac{1}{2} \Rightarrow y_p = \frac{1}{2} e^{-t} \]

\( \alpha = 1 \): Note \( e^{-t} \) and \( te^{-t} \) are solutions to the homogenous equations for this \( \alpha \) so \( y_p = At^2 e^{-t} \)

\[ y = At^2 e^{-t} \]
\[ 2y' = 4Ate^{-t} - 2At^2 e^{-t} \]
\[ y'' = 2Ae^{-t} - 2Ate^{-t} + At^2 e^{-t} - 2At^2 e^{-t} \]

plugging into ODE \( \implies 2Ae^{-t} = e^{-t} \implies A = \frac{1}{2} \)

\[ y_p = \frac{1}{2} t^2 e^{-t} \]

\( \alpha = 2 \):

\[ y_p = Ae^{-t} \Rightarrow Ae^{-t} - 4Ae^{-t} + Ae^{-t} = e^{-t} \Rightarrow A = -\frac{1}{2} \Rightarrow y_p = -\frac{1}{2} e^{-t} \]

using the Method of Undetermined Coefficients.

b) What is the general solution to the ODE for \( \alpha = 1 \).

\[ y_g = y_h + y_p = c_1 e^{-t} + c_2 te^{-t} + \frac{1}{2} t^2 e^{-t} \]

c) Determine the solution to the IVP for \( \alpha = 1 \). (apply initial conditions)

insufficient information.
Problem 3: Method of Undetermined Coefficients Workspace
Problem 4: (15 min) Mass-Spring Oscillator and Phase Plane 25

POINTS Show all Work!

a) Draw a sketch of a Mass-Spring Oscillator with a single mass (1kg) and single spring (k=1) attaching the mass to a wall. Assign y=0 to the rest position of the mass.

b) Derive the second-order ODE equation of motion of the mass for the case when there is NO dampening and NO forcing function.

\[ \sum F = ma \Rightarrow F_{\text{spring}} = ma \Rightarrow -ky = m \cdot y'' \Rightarrow y'' + y = 0 \]

c) Using the assignment \( v = \frac{dy}{dt} \) decompose this single second-order ODE into two first order ODEs.

\[ v' = y'' = -y \]

so:
\[ y' = v \]
\[ v' = -y \]

d) Find the critical points for this system.

\[ y' = 0 \text{ and } v' = 0 \text{ only when } (y, v) = (0, 0) \]

e) Using the two first-order ODEs representing the system, determine an equation for the system trajectories in the \( y - v \) plane.

\[ \frac{dy}{dv} = -\frac{v}{y} \Rightarrow \int ydy = \int -vdv \Rightarrow \frac{y^2}{2} + \frac{v^2}{2} = C \]

f) Draw the trajectory passing through \((y,v)=(1,0)\).

Circle of radius 1 centered at (0,0) traveling in the clockwise direction.
Problem 4: Mass-Spring Oscillator and Phase Plane Workspace
Problem 5: (10 min) Variation of Parameters 15 POINTS  Show all Work!

\[ y'' + y = 1 \]

Use variation of parameters to find the particular solution for this non-homogeneous ODE. Show all steps of the method.

Solve for homogenous solutions (Note \( \alpha = 0 \) case): \( y_1 = \sin(t) \) and \( y_2 = \cos(t) \)

So Variational equations are:

\[
\begin{align*}
y_1' v_1' + y_2' v_2' &= 0 \\
y_1' y_1' + y_2' y_2' &= \frac{g}{a}
\end{align*}
\]

So:

\[
\begin{align*}
sin(t) v_1' + \cos(t) v_2' &= 0 \quad \Rightarrow \quad v_1' = \frac{-v_2' \cos(t)}{\sin(t)} \\
\cos(t) v_1' - \sin(t) v_2' &= 1
\end{align*}
\]

substituting for \( v_1' \) in the second equation we get

\[
\begin{align*}
v_2' \cos^2(t) + v_2' \sin^2(t) &= -\sin(t) \\
v_2' (\cos^2(t) + \sin^2(t)) &= -\sin(t)
\end{align*}
\]

\[ v_2' = -\sin(t) \]

so \( v_1' = \cos(t) \)

integrating both \( v_1' \) and \( v_2' \) we get: \( v_1 = \sin(t) \) and \( v_2 = \cos(t) \)

The particular solution is:

\[ y_p = v_1 y_1 + v_2 y_2 = \sin(t) \sin(t) + \cos(t) \cos(t) = 1 \text{ for every } t. \]
Bonus Problem: (1 min) 10 POINTS  Show all Work!

By simply looking at solutions to previous problems, write down the general solution to this ODE:

\[ y'' + y = 1 + e^{-t} \]

What principle allows you to immediately write down the solution.

Superposition

\[ y_{p1} = \frac{1}{2} e^{-t} \text{ and } y_{p2} = 1 \text{ so } y_p = y_{p1} + y_{p2} = 1 + \frac{1}{2} e^{-t} \]

\[ y_g = y_h + y_p = c_1 \sin(t) + c_2 \cos(t) + 1 + \frac{1}{2} e^{-t} \]