1. (20 pts) Given the matrix
\[ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \]

a) (10 pts) Compute the eigenvalues of A.

**Answer:** First we compute the characteristic polynomial:
\[ \text{char}(A) = \det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = (1 - \lambda)(\lambda^2 + 1). \]

We may compute the determinant by any one of several. Using cofactor expansion over the first row, we compute \( \text{char}(A) = (1 - \lambda)(\lambda^2 + 1) \). This method conveniently gives us the result in partially factored form.

We now factor the characteristic polynomial (equivalently, we find its roots). The first factor is obviously \((1 - \lambda)\), which gives us a root (i.e., an eigenvalue) of \( \lambda = 1 \). The remaining part of the polynomial is \((\lambda^2 + 1)\). We may factor this with the quadratic formula to get:
\[ \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0 \pm \sqrt{0^2 - 4(1)(-1)}}{2(1)} = \pm \frac{\sqrt{4}}{2} = \{i, -i\} \]

Thus, the three eigenvalues are \( \{1, i, -i\} \).

**Check:** We may quickly check this by confirming that the product of the eigenvalues is equal to the determinant of A. As \( \det(A) = 1 \) and the product is \((1)(i)(-i) = 1 \) our work checks.

b) (10 pts) Compute the eigenvectors of A.

**Answer:** For each of the three eigenvalues we compute a corresponding eigenvector by finding a basis for the null space of the matrix \((A - \lambda I)\):

\( \lambda = 1 \): \((A - \lambda I) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \) which we row reduce to \( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). There are pivots in the second and third columns, so \( x_1 \) is a free variable. The second row of the reduced matrix gives us the equation \( x_2 = 0 \) and the third row gives us \( x_3 = 0 \).

Null space vectors are \( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) and an eigenvector is \( v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \).

**Check:** Confirm that \( A v_1 = (1)v_1 \).
\[ \lambda = i : (A - \lambda I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 0 & -i - \frac{-1}{i} \end{pmatrix} \]

which we row reduce:

\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 0 & -i - \frac{-1}{i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{pmatrix} \] (Recall: \( i^2 = -1 \) and \( 1/i = -i \))

There are pivots in the first and second columns, so \( x_3 \) is a free variable. The second row of the reduced matrix gives us the equation \( x_2 - ix_3 = 0 \) or \( x_2 = ix_3 \). The first row gives us the equation \( x_1 = 0 \).

Null space vectors are

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ ix_3 \\ x_3 \end{pmatrix} \]

and an eigenvector is \( v_i = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} \).

Check: Confirm that \( A v_i = (i) v_i \).

\( \lambda = -i \): We could compute \( v_{-i} \) this in the same manner as we computed \( v_i \) above, but a simpler approach is to note that for any pair of complex conjugate eigenvalues of a real matrix, their eigenvectors are conjugates. Thus \( v_{-i} = v_i^* = \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} \).

2. (20 pts) Given the matrix

\[ A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \\ -1 & 2 \end{pmatrix} \]

and the vector \( b = \begin{pmatrix} -3 \\ 8 \\ 3 \end{pmatrix} \) find the best solution to the equation \( Ax = b \) in the least squares sense.

**Answer**: To solve this least squares problem we solve the problem \( A^T Ax = A^T b \).

\[ A^T A \hat{x} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -2 & 2 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -2 \\ -1 & 2 \end{pmatrix} \hat{x} = \begin{pmatrix} 1 & 2 & -1 \\ 8 & 2 & -2 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 8 \\ 3 \end{pmatrix} = A^T b \]

\[ \begin{pmatrix} 6 & -4 \\ -4 & 12 \end{pmatrix} \hat{x} = \begin{pmatrix} 10 \\ -16 \end{pmatrix} \]

We solve this by conventional means to get \( \hat{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

**Check**: Confirm that \( A^T Ax = A^T b \). For a more complete check we confirm that \( (Ax - b) \) is orthogonal to the column space of \( A \), i.e. that \( A^T (Ax - b) = 0 \).
3. (20 pts) Starting with the basis \( \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \) use the Gram-Schmidt algorithm to produce an orthonormal set of vectors which spans the same subspace.

\[
\text{Answer: } v_1 = x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and } w_1 = v_1/\|v_1\| = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}
\]

\[
v_2 = x_2 - \text{proj}_{w_1}(x_2) = x_2 - (x_2 \cdot w_1)w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \sqrt{2} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}
\]

\[
\text{and } w_2 = v_2/\|v_2\| = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

\[
\text{check: we note that } w_1 \cdot w_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \text{ so } w_1 \text{ and } w_2 \text{ are orthogonal}
\]

\[
v_3 = x_3 - \text{proj}_{w_1}(x_3) - \text{proj}_{w_2}(x_3) = x_3 - (x_3 \cdot w_1)w_1 - (x_3 \cdot w_2)w_2
\]

\[
= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \sqrt{2} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

So the first two vectors, \( w_1 \) and \( w_2 \), span the same space as \( x_1, x_2, \) and \( x_3 \).
(We could have seen this before we got started by noting that \( x_3 = (x_1 + x_2)/2 \), so we could have discarded \( x_3 \) at the beginning.)

4. (20 pts) Prove that if \( Q \) is orthogonal and \( A \) is symmetric then \( Q^T A Q \) is symmetric.

\textbf{Proof:} We prove that \( Q^T A Q \) is symmetric by demonstrating that it is its own transpose.

\[
(Q^T A Q)^T = (Q^T A Q)^T \quad \text{(as } Q^T = Q^T) \\
= Q^T A^T Q \quad \text{(transpose behaves as } (CB)^T = B^T C^T) \\
= Q^T Q^T Q \quad \text{(as } Q^T = Q^T) \\
= Q^T A Q \quad \text{(transpose of transpose is the original matrix, } C^{TT} = C) 
\]

Thus the transpose of \( Q^T A Q \) is itself \( Q^T A Q \), so \( Q^T A Q \) is symmetric.

5. (20 pts) State whether each of the following is true in all cases, some cases or no cases. Assume that all matrices are real: ( ALWAYS/SOMETIMES/NEVER)

a) If \( C = A + B \) then \( \det(C) = \det(A) + \det(B) \).
SOMETIMES – False in general, but true in some cases. Obviously true for simple cases such as A=0 or B=0. Less trivial cases also exist. For instance, if 

\[
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},
\]

we let 

\[
B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

and note that this means 

\[
(-2 + d + 4a + ad - 2c - 3b - bc) = (-2) + (ad - bc).
\]

This simplifies to 

\[
4a + d = 3b + 2c,
\]

which has a vast number of solutions, for example 

\[
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}
\]

and 

\[
B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.
\]

b) \(\det(A^T A) \geq 0\), where A is square.

ALWAYS - \(\det(A^T A) = \det(A^T) \det(A) = \det(A) \det(A) = \det(A)^2\)

c) A square matrix with a non-pivot column is onto.

NEVER – With a non-pivot column the null space is non-trivial. From the rank theorem we see that \(\text{rank} = \text{dim(colsp)} < n\), so it is not onto.

d) An invertible matrix is diagonalizable.

SOMETIMES - \[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

is diagonalizable but not invertible, \[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

is invertible but not diagonalizable, \[
\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}
\]

is both diagonalizable and invertible

and \[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

is neither invertible nor diagonalizable.

e) If Q is orthogonal and \(Q^T A Q\) is diagonal then A is symmetric.

ALWAYS – We see that \(A = QDQ^T\), where D is diagonal. As D is symmetric we may use problem 4 to show that A is symmetric.

f) \(\{x^2+1, x^2+x, x^2+x+1\}\) is a basis of \(P_2\). (TRUE/FALSE)

TRUE – This is equivalent to asking if \[
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}
\]

form a basis of \(R^3\). The equivalence is seen by looking at the coefficients of polynomials in \(P_2\), thus

\[
x^2+x = (1)x^2+ (1)x + (0)1 \rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.
\]

g) If \(\det\begin{pmatrix} a & b & 0 \\ c & d & 1 \\ 0 & 5 & 0 \end{pmatrix} = 3\) then \(\det\begin{pmatrix} a & b & 0 \\ c & d & 1 \\ 0 & 7 & 0 \end{pmatrix} = 5\)

NEVER – If we perform a cofactor expansion over the third row on the first matrix we see that \(-5\det\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} = 3\), but the second matrix gives us the contradictory result \(-7\det\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} = 5\)

h) The rank of a square matrix is the number of non-zero columns
SOMETIMES – It is true in the case of \( \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \) (rank 2, with 2 non-zero columns), but false in the case of \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) (rank 1, with 2 non-zero columns). A more accurate statement is that the rank is no greater than the number of non-zero columns.

i) Given: A is not diagonalizable. The eigenvalues of matrix A are distinct. NEVER – A matrix with distinct eigenvalues is always diagonalizable. (see Lay Chap 5.3, Thm 6)

j) Given: the matrix A is orthogonal. A is symmetric.

SOMETIMES – The matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is orthogonal and symmetric, while the matrix \( \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \) is orthogonal but not symmetric.