1) (20 pts) For what values of x does the series \( \sum_{n=2}^{\infty} \frac{(x-3)^n}{5^n} \) converge? Prove it.

This series converges for \(-2 < x < 8\). Prove this by noting that for a given value of x this is a geometric series with the parameter \( r = (x-3)/5 \). Also recall that a geometric series converges if and only if \(|r| < 1\), i.e. if \(|(x-3)/5| < 1\) or \(-2 < x < 8\).

2) (20 pts)
   a) Solve the differential equation \( y' = x \sin(x)/y \).

   This equation is a separable differential equation. Solve by:
   
   \[
   \frac{dy}{dx} = \frac{x \sin(x)}{y} \\
   \int y \, dy = \int x \sin(x) \, dx \\
   y^2/2 = -x \cos(x) + \sin(x) + C \quad \text{(Integrate by parts, u=x, dv=sin(x))}
   \]

   b) If \( y(\pi/2) = 1 \) compute \( y(\pi) \)

   \[
   \frac{1}{2} = 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \cdot \frac{0}{2} - \frac{1}{2} \cdot \frac{1}{2} \\
   \text{So } 1 = 2(1+C) \text{ and } C = -1/2 \\
   \text{Thus } y(\pi) = \sqrt{2 \pi - 1} = 2.298
   \]

3) (15 pts) Use Euler's method with two steps to estimate the value of \( y(\pi) \) if \( y(\pi/2) = 1 \) and \( y' = x \sin(x)/y \).

   \[
   \Delta x = (\pi - \pi/2)/2 = \pi/4 \\
   y(3\pi/4) = y(\pi/2) + \Delta x \ y'(x=\pi/2, \ y=1) \\
   = 1 + \pi/4(\pi/2) = 1 + \pi^2/8 \\
   y(\pi) = y(3\pi/4) + \Delta x \ y'(x=3\pi/4, \ y=1 + \pi^2/8) \\
   = 2.8195
   \]

4) (15 pts) Compute the Taylor series for \( f(x) = x \sin(x) \) about \( x = \pi \) to degree 5.

   This series could either be computed by the straightforward method (compute derivatives up to 5th degree, plug in \( x = \pi \), put into Taylor series formula) or in the following manner:

   Compute Taylor series for \( \sin(x) \) and \( x \) about \( x = \pi \):

   \[
   \sin(x) = -(x-\pi) + (x-\pi)^3/3! - (x-\pi)^5/5! + ... \\
   x = \pi + (x-\pi) \quad \text{(Exactly)}
   \]

   Multiply these series term by term

   \[
   x \sin(x) = (\pi + (x-\pi))(-x-\pi) + (x-\pi)^3/3! - (x-\pi)^5/5! + ... \\
   = -\pi(x-\pi) - (x-\pi)^2 + \pi/3! (x-\pi)^3 + 1/3! (x-\pi)^4 - \pi/5! (x-\pi)^5 - ...
   \]
5) (15 pts) Find \( \lim_{x \to 0} \frac{\sin(x)}{e^x - 1} \).

First try to naively plug in the value \( x=0 \) to get the indeterminate form 0/0.

Now use L'Hopital's rule, differentiating \( \sin(x) \) to get \( \cos(x) \) and differentiating \( e^x - 1 \) to get \( e^x \), so
\[
\lim_{x \to 0} \frac{\sin(x)}{e^x - 1} = \lim_{x \to 0} \frac{\cos(x)}{e^x} = \frac{1}{1} = 1
\]

6) (15 pts) True or False

a) Estimating \( \int_0^\pi \sin(x) \, dx \) using Trapezoid method will yield an answer which is larger than the correct answer. FALSE. Drawing a picture you see that \( \sin(x) \) is concave downward in this region, so the trapezoid method will always yield an answer which is smaller than the correct answer.

\[ \int_0^\pi \sin(x) \, dx \]

b) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) is absolutely convergent. FALSE, taking absolute values of each term gives the harmonic series, \( \sum \frac{1}{n} \), which is divergent.

c) \( \int \frac{dx}{1 + e^x} = \log \left( \frac{e^x}{1 + e^x} \right) + C \) TRUE - The easiest way to check this is not to integrate the left hand side, but to differentiate the right hand side.

d) \( \{x=\sin(t), y=\cosh(t)\} \) describes a parabola. FALSE - This describes the hyperbola, which can be seen by recalling that \( \cosh(t)^2 - \sinh(t)^2 = 1 \), ie \( y^2 - x^2 = 1 \).

e) \( \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{27} \). FALSE. Recall that the sum of the geometric series starting at \( n=0 \) is \( \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \) and that dropping the first 3 terms drops \( 1+1/3+1/9 \), so \( \frac{3}{2} - (1+1/3+1/9) = 1/18 \) is the sum.