

1. **True or False** (with a very brief explanation or counterexample):

- (a) The union of a countable family of compact sets is compact.
- (b) The union of any family of open sets is open
- (c) If (A, d) is a subspace of a complete metric space (M, d) and U is an open set in (A, d) , then there must be some open set V in (M, d) such that $U = V \cap A$.
- (d) If $f : X \rightarrow Y$ is continuous and K is compact, then $f^{-1}(K)$ is compact.
- (e) If a complete metric space is totally bounded, then it is compact.
- (f) The boundary of the union of two open sets is contained in the union of their boundaries.
- (g) If A, B are pathwise connected open sets, then $A \cap B$ is connected.
- (h) If A is connected and compact, then A is pathwise connected.
- (i) For $f : \mathbb{R}^n \rightarrow \mathbb{R}$, if all directional derivatives exist at each point $x \in \mathbb{R}^n$, then f is continuous.
- (j) If $x \mapsto f(f(x))$ is a contraction mapping: $\mathbb{R} \rightarrow \mathbb{R}$, then f must be continuous.
- (k) If $\{f_n\}$ is an equicontinuous family of continuously differentiable functions: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ and $f_n \rightarrow \bar{f}$ uniformly, then \bar{f} is differentiable.

2. For any 4 of the following, give a correct **definition**:

accumulation point, boundary, Cantor set, Cartesian product (of 2 metric spaces), Cauchy sequence, closed set, compact, complete, connected, contraction mapping, $C(X)$, equicontinuous, norm, fixed point, sequentially compact, subcovering, totally bounded, uniformly continuous, uniformly convergent

3. For any 3 of the following, give a precise **statement**:

Arzela-Ascoli Theorem, Bolzano-Weierstrass Theorem, Contraction Mapping Principle, completeness of $C(X)$, Heine-Borel Theorem, Intermediate Value Theorem, Inverse Function Theorem, Mean Value Theorem, subsubsequence lemma

4. For each of these: give an example [indicating how one knows that it is an example].

- (a) a set \mathcal{A} which is not open, is not closed, is not bounded.
- (b) an uncountable closed set with empty interior
- (c) a connected set which is not pathwise connected
- (d) a contraction mapping with no fixed point
- (e) a bounded closed set in $C([0, 1])$ which is not compact.

5. Look at all the HW problems which have been assigned; do any 3 of them.

6. **Extra Credit:** A function $f : X \rightarrow \mathbb{R}$ is called lower semi-continuous (lsc) if:

For every $\bar{x} \in X$, for every $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon, \bar{x}) > 0$ such that

$$x \in X, d(x, \bar{x}) \leq \delta \quad \Rightarrow \quad f(\bar{x}) - f(x) \leq \varepsilon.$$

(a) Give an example of a function f which is lsc but is not continuous.

(b) **Prove:** *If $f : X \rightarrow \mathbb{R}$ is lsc and X is compact, then f attains its minimum — i.e., there is some $\bar{x} \in X$ such that $f(x) \geq f(\bar{x})$ for every $x \in X$.*