As the vector $D^{-1}\mathbf{b}$ has coordinates $\alpha^2 - 2\alpha r + r(r-1), r = 0, 1, \ldots, R$,

$$U^T D^{-1}\mathbf{b} = \begin{pmatrix} \sum_0^R p_r(\alpha)(\alpha^2 - 2\alpha r + r(r-1)) \\ \alpha^{-1/2}\sum_0^R p_r(\alpha)(\alpha - r)(\alpha^2 - 2\alpha r + r(r-1)) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(\alpha - R)p_R(\alpha) \\ \alpha^{1/2}[(\alpha-R)^2 + R]p_R(\alpha) \end{pmatrix}.$$

Also,

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - U^T D^{-1} U\right]^{-1} = \frac{1}{\sum_{R+1} p_r(\alpha) d}\begin{pmatrix} 1 + (R - \alpha + 1)\theta & \sqrt{\alpha}\theta \\ \sqrt{\alpha}\theta & 1 \end{pmatrix}.$$

Therefore, $\Sigma^{-1}\mathbf{b}$ has coordinates given in (18), and

$$\mathbf{b}^T\mathbf{w} = \sum_0^R p_r(\alpha)[\alpha^2 - 2\alpha r + r(r-1)]^2$$

$$+\sum_0^R p_r(\alpha)(\alpha - r)[\alpha^2 - 2\alpha r + r(r-1)]\theta\frac{[(\alpha-R)^2 + \alpha\theta(\alpha - R) + R]}{d}$$

$$+\sum_0^R p_r(\alpha)(\alpha^2 - 2\alpha r + r(r-1))\frac{(\alpha - R + \alpha\theta)}{d},$$

so that (19) holds. Differentiating with regard to α, one obtains

$$\frac{d}{d\alpha}\mathbf{b}^T\Sigma^{-1}\mathbf{b} = 4\alpha\sum_0^{R-1} p_r(\alpha) - 2\alpha R p_R(\alpha)$$

$$+p_R(\alpha)\frac{(\alpha - R + \alpha\theta)[(\alpha - R)^2 + \alpha\theta(\alpha - R) + R]}{d}$$

$$\times\left[1 + \frac{\alpha(1+\theta)}{\alpha - R + \alpha\theta} + \frac{2\alpha(\alpha - R + \alpha\theta)}{(\alpha-R)^2 + \alpha\theta(\alpha - R) + R} - \frac{\alpha - R + \alpha\theta}{d}\right].$$

When $\alpha = R$, the known asymptotic expansion for the incomplete gamma-function shows that $p_R(R) \sim (2\pi R)^{-1/2}, \theta \sim (\pi R/2)^{-1/2}$, and $\sum_0^{R-1} p_r(R) = \int_R^\infty e^{-u}u^{R-1}\,du/\Gamma(R) \sim 1/2$, so that

$$\frac{d}{d\alpha}\mathbf{b}^T\Sigma^{-1}\mathbf{b}\Big|_{\alpha=R} = -\frac{(\pi^2 - 4\pi + 8)}{\sqrt{2\pi}(\pi - 2)\pi}R^{3/2} + O(R).$$

Therefore for large R, $\mathbf{b}^T\Sigma^{-1}\mathbf{b}$ decreases at $\alpha = R$. Since

$$\frac{\theta(\alpha - R + \alpha\theta)[(\alpha-R)^2 + \alpha\theta(\alpha - R) + R]}{d} \le 2\alpha(1+\theta),$$

for sufficiently large R,

$$\mathbf{b}^T\Sigma^{-1}\mathbf{b} \le 2R^2.$$

Thus, the maximum of this function cannot exceed $2R^2$. On the other hand when $\alpha^\star = \beta R$ with $0 < \beta < 1, p_R(\beta R) \sim (2\pi R)^{-1/2}e^{R(1-\beta)}\beta^R, \theta \sim (1-\beta)/\beta$, and $\sum_0^{R-1} p_r(\beta R) \sim 1$. Therefore, $\mathbf{b}^T\Sigma^{-1}\mathbf{b} \sim 2\beta^2 R^2$, and the limit in (21) cannot be smaller than $2\beta^2$.