Advice:

Note that this writeup consists of complete sentences.

I don't know about you, but if I cannot explain my thoughts in coherent sentences, I know that I don't understand the stuff.

Try this for yourself: Pick a homework problem and write the solution in complete sentences with proper grammar, punctuation, spelling, etc., like the solutions that you are about to read. Can you? You will be doing yourself a big favor if you do.

This is ***an excellent way*** *to find gaps in your knowledge. You will find that once you know where the gaps are, filling them is not too difficult.*

1. Brine of concentration of c_{in} is entering a mixing tank at the volume rate of Q_{in} and after thoroughly mixing with the contents of the tank, exits at a different volume rate, Q_{out}. Assume c_{in}, Q_{in} and Q_{out} are constants. Develop a differential equation for the concentration $c(t)$ of salt in the tank. Assume that the tank contains an initial volume V_0 of water. Explain your work.

   Solution: The volume of brine in the tank increases at the rate of $Q_{\text{in}} - Q_{\text{out}}$ therefore $V(t) = V_0 + (Q_{\text{in}} - Q_{\text{out}})t$.

   Let $m(t)$ be the mass of the salt in the tank and $c(t)$ be its concentration. We have $m(t) = V(t)c(t) = \big[V_0 + (Q_{\text{in}} - Q_{\text{out}})t\big]c(t)$.

   Salt enters the tank at the rate of $Q_{\text{in}}c_{\text{in}}$ (mass per unit time) and leaves the tank at the rate of $Q_{\text{out}}c(t)$. The mass of salt is conserved:

   $$m'(t) = Q_{\text{in}}c_{\text{in}} - Q_{\text{out}}c(t).$$

   Substitute for $m(t)$ from what was calculated above:

   $$\Big(\big[V_0 + (Q_{\text{in}} - Q_{\text{out}})t\big]c(t)\Big)' = Q_{\text{in}}c_{\text{in}} - Q_{\text{out}}c(t).$$

2. Solve the initial value problem $y' = \dfrac{1-y^2}{3xy}$, $y(1) = 3$. Put in a word or two saying what you are doing.

   Solution method #1: Separating variables

   Separate the variables:

   $$\frac{y\,dy}{1-y^2} = \frac{1}{3x},$$

   multiply both sides by -2,

   $$\frac{-2y\,dy}{1-y^2} = -\frac{2}{3x},$$

and integrate:

$$\ln|1-y^2| = -\frac{1}{3}\ln|x| + C.$$

Then rearrange as:

$$\ln|1-y^2| + \frac{2}{3}\ln|x| = C$$

and:

$$\ln|1-y^2|\ln|x|^{\frac{2}{3}} = C.$$

Therefore:

$$\ln\left|(1-y^2)x^{\frac{2}{3}}\right| = C,$$

and consequently:

$$\left|(1-y^2)x^{\frac{2}{3}}\right| = e^C$$

whence $(1-y^2)x^{\frac{2}{3}} = K$. Applying the initial condition $y(1) = 3$ we get K = -8. therefore $(1-y^2) = -8x^{-\frac{2}{3}}$ and $y = \pm\sqrt{1+8x^{-\frac{2}{3}}}$. We observe that only the plus sign is acceptable because the expression with minus sign fails to satisfy the initial condition. We conclude that:

$$y = \sqrt{1+8x^{-\frac{2}{3}}}.$$

Solution method #2: Solving as a Bernoulli equation

This equation is also a Bernoulli equation as it can be rearranged as $y' = (1-y^2)/(3xy) = 1/(3xy) - y/(3x)$, therefore:

$$y' + \frac{1}{3x}y = \frac{1}{3x}y^{-1}.$$

Following the general method for solving Bernoulli equations, we multiply both sided by y and get

$$yy' + \frac{1}{3x}y^2 = \frac{1}{3x},$$

then let $y^2 = u$. Therefore $2yy' = u'$ thus the equation reduces to

$$\frac{1}{2}u' + \frac{1}{3x}u = \frac{1}{3x},$$

which is a first order linear equation. We put it in the standard form:

$$u' + \frac{2}{3x}u = \frac{2}{3x}.$$

The integrating factor is

$$\mu = e^{\int \frac{2}{3x}\,dx} = e^{\frac{2}{3}\ln x} = e^{\ln x^{\frac{2}{3}}} = x^{\frac{2}{3}}.$$

Multiplying the equation through by μ we get:

$$x^{\frac{2}{3}}u' + \frac{2}{3}x^{-\frac{1}{3}}u = \frac{2}{3}x^{-\frac{1}{3}}$$

which collapses to:

$$\left(x^{\frac{2}{3}}u\right)' = \frac{2}{3}x^{-\frac{1}{3}}.$$

Upon integrating this we get $x^{\frac{2}{3}}u = x^{\frac{2}{3}} + C$ whence $u = 1 + Cx^{-\frac{2}{3}}$ and $y = \pm\sqrt{1 + Cx^{-\frac{2}{3}}}$. The constant of integration, C, is determined by applying the initial condition which says $3 = \pm\sqrt{1+C}$ from which we see that the negative sign is unacceptable. We keep the positive sign and solve for C and we get $C = 8$. Therefore the solution is $y = \sqrt{1 + 8x^{-\frac{2}{3}}}$.

3. Find the general solution of the differential equation $xy' = \dfrac{1}{x^3} - 2y$. Put in a word or two saying what you are doing.

   Solution: This is a linear first order equation. We put it in the standard form:

$$y' + \frac{2}{x}y = \frac{1}{x^4}.$$

   The integrating factor is:

$$\mu = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = e^{\ln x^2} = x^2.$$

   We multiply the equation through by μ:

$$x^2y' + 2xy = \frac{1}{x^2}.$$

   This collapses to

$$(x^2y)' = \frac{1}{x^2}$$

   which is easily integrated:

$$x^2y = -\frac{1}{x} + C,$$

   whence

$$y = -\frac{1}{x^3} + \frac{C}{x^2}.$$

4. Find the general solution of the differential equation $y' = \dfrac{2y}{x} - x^2y^2$. Put in a word or two saying what you are doing.

   Solution: This a Bernoulli equation. We put it in the standard form:

$$y' - \frac{2}{x}y = -x^2y^2,$$

   then we divide both sides by y^2:

$$y^{-2}y' - \frac{2}{x}y^{-1} = -x^2.$$

   Following the usual procedure, we let $u = y^{-1}$ and observe that $u' = -y^{-2}y'$. Thus the equation reduces to $-u' - (2/x)u = -x^2$. This is a first order linear equation. We put it in the standard form:

$$u' + \frac{2}{x}u = x^2$$

and compute the integrating factor:

$$\mu = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = e^{\ln x^2} = x^2.$$

We multiply the equation by μ:

$$x^2u' + 2xu = x^4$$

then contract it to $(x^2u)' = x^4$ whence $x^2u = x^5/5 + c$. Consequently $u = x^3/5 + c/x^2$. Since $y = 1/u$, we conclude that:

$$y = \frac{1}{\frac{x^3}{5} + \frac{c}{x^2}} = \frac{x^2}{\frac{x^5}{5} + c} = \frac{5x^2}{x^5 + C}.$$

5. Consider the initial value problem $y' = 1 + y^2$, $y(0) = 0$. Apply Euler's method with step size $h = 0.2$ to find an approximate value for $y(1)$. Put in a word here and there to guide me in reading through your calculations.

Solution: We let $f(x, y) = 1 + y^2$. Then Euler's algorithm generates solution points (x_n, y_n) through

$$x_{n+1} = x_n + h, \quad \text{and} \quad y_{n+1} = y_n + hf(x_n, y_n),$$

where h is the stepsize ($h = 0.2$ in our case) and (x_0, y_0) coincides with the initial condition, that is, $x_0 = 0$, $y_0 = 0$. Therefore $x_1 = x_0 + 0.2 = 0.2$ and $y_1 = y_0 + hf(x_0, y_0) = 0 + 0.2 \times (1 + 0^2) = 0.2$. Continuing in that way we obtain:

n	x_n	y_n
0	0	0
1	0.2	0.2
2	0.4	0.408
3	0.6	0.6412928
4	0.8	0.9235440910
5	1.0	1.294130828695

6. In 1970 the population of alligators in the Kennedy Space Center was estimated to be around 300. In 1980 the population had grown to 1500. Suppose that the population grows according to the Malthusian model, that is, $dp/dt = \mu p$ for some constant μ. Solve the differential equation and use it to determine μ from the given data. What will the population be in the year 2010? Explain your work.

Solution: The differential equation $dp/dt = \mu p$ is separable since it may be written as $(1/p)\,dp = \mu\,dt$. Thus $\ln p = \mu t + c$. We don't need absolute values here because both the population p and the time t are non-negative quantities. If the initial population is p_0, we see that $c = \ln p_0$. Then the solution becomes $\ln p - \ln p_0 = \mu t$, that is, $\ln(p/p_0) = \mu t$ whence $p(t) = p_0 e^{\mu t}$.

We measure the time in years beginning with the year 1970 as $t = 0$. According to the problem's data we have $p(0) = 300$ and $p(10) = 1500$.

Plugging this into the solution obtained above results in the equations in the two unknowns p_0 and μ:

$$300 = p_0 \quad \text{and} \quad 1500 = p_0 e^{10\mu}.$$

We solve this to get $p_0 = 300$ and and $\mu = (\ln 5)/10$. To find the population in the year 2010 we take $t = 2010 - 1970 = 40$:

$$p(40) = 300e^{\frac{50 \ln 5}{10}} = 300e^{4 \ln 5} = 300e^{\ln 5^4} = 300 \times 5^4 = 300 \times 625 = 187,500.$$