Predictor-corrector methods for sufficient linear complementarity problems in a wide neighborhood of the central path*

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Two predictor-corrector methods of order $m = \Omega(\log n)$ are proposed for solving sufficient linear complementarity problems. The methods produce a sequence of iterates in the $\mathcal{N}_\infty^-$ neighborhood of the central path. The first method requires an explicit upper bound κ of the handicap of the problem while the second method does not. Both methods have $O((1+\kappa)^{1+1/m}\sqrt{n}L)$ iteration complexity. They are superlinearly convergent of order $m+1$ for nondegenerate problems and of order $(m+1)/2$ for degenerate problems. The cost of implementing one iteration is at most $O(n^3)$ arithmetic operations.

Keywords: linear complementarity, interior-point, path-following, predictor-corrector

NOTE: in this report we correct an error from the paper with the same title published in Optimization Methods and Software, 20, 1 (2005), 145–168, (see Erratum at http://www.math.umbc.edu/~potra/erratum2.pdf**).**

1 Introduction

Primal-dual interior-point methods play an important role in modern mathematical programming. They have been widely used to obtain strong theoretical results and they have been successfully implemented in software packages for solving linear programming (LP), quadratic programming (QP), semidefinite programming (SDP), and many other problems. For an excellent analysis of primal-dual interior-point methods and their implementation see the excellent monograph of Stephen Wright [24].

The MTY predictor-corrector algorithm proposed by Mizuno, Todd and Ye [12] is a typical representative of a primal-dual interior-point method for LP. It has $O(\sqrt{n}L)$ iteration complexity, which is the best iteration complexity obtained so far for any interior-point method. Moreover, the duality gap of the sequence generated by the MTY algorithm converges to zero quadratically [27]. The MTY algorithm was the first algorithm for LP having both polynomial complexity and

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superlinear convergence. Ji, Potra and Huang [8] generalized the MTY algorithm to monotone linear complementarity problems (LCP). The method has $O(\sqrt{n}L)$ iteration complexity and superlinear convergence, under the assumption that the LCP is non-degenerate and the iteration sequence converges. It turns out that these assumptions are not restrictive. Thus the convergence of the iteration sequence follows from a general result of [4] and, according to [13], the nondegeneracy (i.e. the existence of a strict complementarity solution) is a necessary condition for superlinear convergence. We note that in [13] it is shown that in the degenerate case a large class of first order interior-point methods, which contains the MTY algorithm, can only achieve linear convergence with factor at most .25. A direct proof of the superlinear convergence of the MTY algorithm for nondegenerate LCPs, without using the convergence of the iteration sequence, is contained in [26]. The MTY algorithm operates in a l_2 neighborhood of the central path. It is well known however that primal-dual interior-point methods have a better practical performance in a wider neighborhood of the central path. The most efficient primal-dual interior-point methods operate in the so called $\mathcal{N}_\infty^-$ neighborhood, to be defined later in the present paper. Unfortunately, the iteration complexity of the predictor-corrector methods that use wide neighborhoods are worse than the complexity of the corresponding methods for small neighborhoods. By using the analysis of Anstreicher and Bosch [3] it follows that the iteration complexity of a straightforward implementation of a predictor-corrector method in the large neighborhood of the central path would be $O(n^{3/2}L)$. Gonzaga [5] proposed a predictor-corrector method using the $\mathcal{N}_\infty$ neighborhood of the central path that has $O(nL)$ iteration complexity. In contrast with the MTY algorithm that uses a predictor step followed by a corrector step at each iteration, Gonzaga's algorithm uses a predictor step followed by a variable number of corrector steps at each iteration. There are no sharp estimates on the number of the corrector steps that are needed at each iteration. However by using a very elegant analysis Gonzaga was able to prove that his algorithm needs at most $O(nL)$ predictor and corrector steps. The results of [3, 5] show that it is more difficult to develop and analyze MTY type of predictor-corrector methods in large neighborhoods. The best iteration complexity achieved by any known interior-point method in the large neighborhood using first order information is $O(nL)$. As shown in [6, 28], the iteration complexity can be reduced to $O(\sqrt{n}L)$ by using higher order information. However the algorithms presented in those papers are not of MTY type and it appears that they are not superlinearly convergent. A higher order algorithm of MTY type in the $\mathcal{N}_\infty^-$ neighborhood with $O(\sqrt{n}L)$ complexity and superlinear convergence has recently been proposed in [15].

The existence of a central path is crucial for interior-point methods. An important result of the 1991 monograph of Kojima et al. [9] shows that the central path exists for any P_* linear complementarity problem, provided that the relative interior of its feasible set is nonempty. We recall that every P_* linear complementarity problem is a $P_*(\kappa)$ problem for some $\kappa \geq 0$, i.e.

$$P_* = \cup_{\kappa \geq 0} P_*(\kappa) .$$

The class $P_*(0)$ coincides with the class of monotone linear complementarity problems, and $0 \leq \kappa_1 \leq \kappa_2$ implies $P_*(\kappa_1) \subset P_*(\kappa_2)$. A surprising result of Väliaho [22] from 1996 showed that the class of P_* matrices coincides with the class of sufficient matrices. Therefore, the interior-point methods of [9] can solve any sufficient linear complementarity problem. Of course, the computational complexity of the algorithm depends on the parameter κ. The best known iteration complexity of an interior-point method for a $P_*(\kappa)$ problem is $O((1+\kappa)\sqrt{n}L)$. No superlinear complexity results were given for the interior-point methods of [9].

In 1995 Miao [10] extended the MTY predictor-corrector method for $P_*(\kappa)$ linear complementarity problems. His algorithm uses the l_2 neighborhood of the central path, has $O((1+\kappa)\sqrt{n}L)$ iteration complexity, and is quadratically convergent for nondegenerate problems. However, the constant κ is explicitly used in the construction of the algorithm. Or, it is well known that this constant is very difficult to estimate for many sufficient linear complementarity problems. The predictor-corrector methods described in [16] improve on Miao's algorithm in several ways. First, the algorithms do not depend on the constant κ, so that the same algorithm is used for any sufficient complementarity problem. Second, the neighborhoods used by the algorithms of [16] are slightly larger than those considered in [10]. Third, by employing a higher order predictor, the algorithms of [16] may attain arbitrarily high orders of convergence on nondegenerate problems. Finally, by using the "fast-safe-improve" strategy of Wright and Zhang [25], the algorithms of [16] require asymptotically only one matrix factorization per iteration, while Miao's algorithm, as well as the original MTY algorithm, require two matrix factorizations at every iteration. While the algorithms of [16] do not depend on the constant κ, their computational complexity does: if the problem is a $P_*(\kappa)$ linear complementarity problem they terminate in at most $O((1+\kappa)\sqrt{n}L)$ iterations. The predictor-corrector algorithms presented in [17, 18] are superlinearly convergent even for degenerate problems. More precisely the Q-order of convergence of the complementarity gap is 2 for nondegenerate problems and 1.25 for degenerate problems. The algorithms of [17, 18] are first order methods that do not belong to the class of interior-point methods considered in [13], so that the fact that they are superlinearly convergent for degenerate problems does not contradict the result of that paper. In the degenerate case superlinear convergence is achieved by employing an idea of Mizuno [11] which consists in identifying indices for which strict complementarity does not hold, which is possible when the complementarity gap is small enough, and by using an extra backsolve in order to accelerate the convergence of the corresponding variables.

Predictor-corrector algorithms with arbitrarily high order of convergence for degenerate sufficient linear complementarity problems were given in [20]. The algorithms depend on the constant κ, use a l_2 neighborhood of the central path and, as shown in [19], have $O((1+\kappa)\sqrt{n}L)$ iteration complexity for $P_*(\kappa)$ linear complementarity problems. A general local analysis of higher order predictor-corrector methods in a l_2 neighborhood for degenerate sufficient linear complementarity problems is given by Zhao and Sun [29], who also propose a new algorithm that does not need a corrector step. The latter algorithm does not follow the traditional central path. Instead a new analytic path is used at each iteration. No complexity results are given.

All the above mentioned interior-point methods for sufficient linear complementarity problem use small neighborhoods of the central path. In a recent monograph [7], Peng, Roos and Terlaky propose the use of larger neighborhoods of the central path defined by means of self-regular functions. They propose an interior-point algorithm for solving a class of $P_*(\kappa)$ nonlinear complementarity problems based on such larger neighborhoods. Their algorithm does not depend on κ. They establish the complexity of the algorithm in terms of n only, by tacitly assuming that κ is a finite constant. By analyzing the proof of Theorem 4.4.10 of the monograph it follows that the iteration complexity of their algorithm (with $q = \log n$) when applied to a $P_*(\kappa)$ linear complementarity problem is $O((1+\kappa)\log n\sqrt{n}L)$. Since at each main iteration of their algorithm the complementarity gap is reduced by a given constant, their algorithm is only linearly convergent. A superlinear interior-point algorithm for sufficient linear complementarity problems in the $\mathcal{N}_\infty^-$ neighborhood has been proposed by Stoer [21]. This algorithm is an adaptation of the second algorithm of [29] for the large neighborhood. No complexity results are proved.

In the present paper we will propose several predictor-corrector methods for sufficient horizontal linear complementarity problems (HLCP) in the $\mathcal{N}_\infty^-$ neighborhood of the central path that extends the algorithm of [15]. HLCP is a slight generalization of the standard linear complementarity problem (LCP). As shown by Anitescu et al. [1], different variants of the $P_*(\kappa)$ linear complementarity problem, including LCP, HLCP, mixed LCP and geometric LCP are equivalent in the sense that any complexity or superlinear convergence result proved for any of the formulations is valid for all formulations. We choose to work on $P_*(\kappa)$ HLCP because of its symmetry. We will start by describing a first order predictor-corrector method for the $P_*(\kappa)$ HLCP that depends explicitly on the constant κ. We will prove that the algorithm has $O((1+\kappa)nL)$ iteration complexity for general $P_*(\kappa)$ problems and is quadratically convergent for nondegenerate problems. If κ is not known, we will propose a first order predictor-corrector method with $O((1+\chi)nL)$ iteration complexity, where χ is the handicap of the sufficient linear complementarity problem, i.e. the smallest $\kappa \geq 0$ for which the problem is a $P_*(\kappa)$ problem. Both algorithms belong to the class of interior-point methods considered in [13] so that they are not superlinearly convergent on degenerate problems. Of course we could modify them using the ideas of [11], [17], [18] to obtain an algorithm with Q-order 1.25 on degenerate problems. However this would depend on an efficient identification of the indices for which strict complementarity does not hold and would require an extra backsolve. Or with an extra backsolve we can construct a second order method that has Q-order 1.5 on degenerate problems and does not depend on identification of the indices for which strict complementarity fails. More generally, by using a predictor of order $m \geq 2$ we obtain algorithms with $O((1+\kappa)^{1+1/m}n^{1/2+1/(1+m)}L)$ iteration complexity if κ is given and $O((1+\chi)^{1+1/m}n^{1/2+1/(1+m)}L)$ iteration complexity if κ is not given. The higher order methods are superlinearly convergent even for degenerate $P_*(\kappa)$ HLCP. More precisely the Q-order of convergence of the complementarity gap is $m+1$ for nondegenerate problems and $(m+1)/2$ for degenerate problems. By choosing $m = \Omega(\log n)$, the iteration complexity of our algorithms reduces to $O((1+\kappa)^{1+1/m}\sqrt{n}L)$ and to $O((1+\chi)^{1+1/m}\sqrt{n}L)$, respectively. The results of the present paper represent a nontrivial generalization of the results of [15] since some of the proof techniques do not carry over from the monotone case, and since the algorithms of [15] had to be modified in such a way that they do not depend on κ. To our knowledge the algorithms presented in this paper have the lowest complexity bounds of any interior-point methods for sufficient linear complementarity problems acting in the $\mathcal{N}_\infty^-$ neighborhood of the central path. Moreover they are predictor-corrector methods that are superlinearly convergent even for degenerate problems.

Conventions. We denote by $I\!N$ the set of all nonnegative integers. $I\!R$, $I\!R_+$, $I\!R_{++}$ denote the set of real, nonnegative real, and positive real numbers respectively.

Given a vector x, the corresponding upper case symbol X denotes the diagonal matrix defined by the vector. We denote component-wise operations on vectors by the usual notations for real numbers. Thus, given two vectors u, v of the same dimension, uv, u/v, etc. will denote the vectors with components u_iv_i, u_i/v_i, etc, that is, $uv \equiv Uv$. Also if f is a scalar function and v is a vector, then $f(v)$ denotes the vector with components $f(v_i)$. For example if $v \in I\!R^n_+$, then $\sqrt{v}$ denotes the vector with components $\sqrt{v_i}$, $|v|$ denotes the vector with components $|v_i|$, and 1-v denotes the vector with components $1-v_i$. Traditionally the vector $1-v$ is written as $e-v$, where e is the vector of all ones. The inequalities $v \geq 0$ and $v > 0$ are also understood componentwise. $[v]^-$ denotes the negative part of the vector v, $[v]^- = \max\{-v,\, 0\}$.

If $x, s \in I\!R^n$, then the vector $z \in I\!R^{2n}$ obtained by concatenating x and s will be denoted by $\lceil x,\, s \rfloor$, i.e.,

$$z = \lceil x,\, s \rfloor = \begin{bmatrix} x \\ s \end{bmatrix} = \left[\, x^T,\, s^T \,\right]^T. \tag{1.1}$$

Throughout this paper the mean value of xs will be denoted by

$$\mu(z) = \frac{x^Ts}{n}. \tag{1.2}$$

If $\|\,.\,\|$ is a vector norm on $I\!R^n$, and A is a matrix, then the operator norm induced by $\|\,.\,\|$ is defined in the usual manner by $\|\,A\,\| = \max\{\|\,Ax\,\|\ ;\ \|\,x\,\| = 1\}$.

We use the notations $O(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$, and $o(\cdot)$ in the standard way: If $\{\tau_k\}$ is a sequence of positive numbers tending to 0 or ∞, and $\{x^k\}$ is a sequence of vectors, then $x^k = O(\tau_k)$ means that there is a constant ϑ such that for every $k \in I\!N$, $\left\|\,x^k\,\right\| \leq \vartheta\tau_k$; if $|x^k| > 0$, $x^k = \Omega(\tau_k)$ means that $(x^k)^{-1} = O(1/\tau_k)$. If we have both $x^k = O(\tau_k)$ and $x^k = \Omega(\tau_k)$, we write $x^k = \Theta(\tau_k)$. Finally, $x^k = o(\tau_k)$ means that $\lim_{k\to\infty}\left\|\,x^k\,\right\|/\tau_k = 0$. For any real number ρ, $\lceil\, \rho\, \rceil$ denotes the smallest integer greater or equal to ρ.

2 The $P_*(\kappa)$ horizontal linear complementarity problem

Given two matrices $Q, R \in \mathbb{R}^{n\times n}$, and a vector $b \in \mathbb{R}^n$, the horizontal linear complementarity problem (HLCP) consists in finding a pair of vectors $z = \lceil x, s \rfloor$ such that

$$\begin{aligned} xs &= 0 \\ Qx + Rs &= b \\ x, s &\geq 0. \end{aligned} \tag{2.1}$$

The standard (monotone) linear complementarity problem (SLCP or simply LCP) is obtained by taking $R = -I$, and Q positive semidefinite. Let $\kappa \geq 0$ be a given constant. We say that (2.1) is a $P_*(\kappa)$ HLCP if

$$Qu + Rv = 0 \text{ implies } (1+4\kappa)\sum_{i\in\mathcal{I}^+} u_i v_i + \sum_{i\in\mathcal{I}^-} u_i v_i \geq 0, \text{ for any } u, v \in \mathbb{R}^n$$

where $\mathcal{I}^+ = \{i : u_i v_i > 0\}$, $\mathcal{I}^- = \{i : u_i v_i < 0\}$. If the above condition is satisfied we say that (Q, R) is a $P_*(\kappa)$ pair and we write $(Q, R) \in P_*(\kappa)$. If (Q, R) belongs to the class

$$P_* = \cup_{\kappa\geq 0} P_*(\kappa)$$

then we say that (2.1) is a P_* HLCP. In case $R = -I$, $(Q, -I)$ is a $P_*(\kappa)$ pair if and only if Q is a $P_*(\kappa)$ matrix in the sense that:

$$(1+4\kappa)\sum_{i\in\hat{\mathcal{I}}^+} x_i[Qx]_i + \sum_{i\in\hat{\mathcal{I}}^-} x_i[Qx]_i \geq 0, \quad \forall x \in \mathbb{R}^n$$

where $\hat{\mathcal{I}}^+ = \{i : x_i[Qx]_i > 0\}$, $\hat{\mathcal{I}}^- = \{i : x_i[Qx]_i < 0\}$. Problem (2.1) is then called a $P_*(\kappa)$ LCP and it is extensively discussed in [9].

A matrix Q is called column sufficient if $x(Qx) \leq 0$ implies $x(Qx) = 0$, and row sufficient if $x(Q^T x) \leq 0$ implies $x(Q^T x) = 0$. A matrix that is both row sufficient and column sufficient is called a sufficient matrix. Väliaho's result [22] states that a matrix is sufficient if and only if it is a $P_*(\kappa)$ matrix for some $\kappa \geq 0$. By extension, a P_* HLCP will be called a sufficient HLCP and a P_* pair will be called a sufficient pair. The handicap of a sufficient pair (Q, R) is defined as

$$\chi(Q, R) := \min\{\kappa \;:\; \kappa \geq 0, (Q, R) \in P_*(\kappa)\}. \tag{2.2}$$

A general expression for the handicap of a sufficient matrix and a method for determining it is described in [23].

We denote the set of all feasible points of HLCP by

$$\mathcal{F} = \{z = \lceil x, s \rfloor \in \mathbb{R}^{2n}_+ \;:\; Qx + Rs = b\},$$

and its solution set by

$$\mathcal{F}^* = \{z^* = \lceil x^*, s^* \rfloor \in \mathcal{F} \;:\; x^* s^* = 0\}.$$

The relative interior of $\mathcal{F}$, which is also known as the set of strictly feasible points or the set of interior points, is given by

$$\mathcal{F}^0 = \mathcal{F} \bigcap I\!R^{2n}_{++} .$$

It is known (see, for example, [9]) that if $\mathcal{F}^0$ is nonempty, then the nonlinear system,

$$\begin{aligned} xs &= \tau e \\ Qx + Rs &= b \end{aligned}$$

has a unique positive solution for any $\tau > 0$. The set of all such solutions defines the central path $\mathcal{C}$ of the HLCP, that is,

$$\mathcal{C} = \{ z \in I\!R^{2n}_{++} \ : \ F_\tau(z) = 0, \, \tau > 0 \} ,$$

where

$$F_\tau(z) = \left[\begin{array}{c} xs - \tau e \\ Qx + Rs - b \end{array} \right] .$$

If $F_\tau(z) = 0$, then it is easy to see that $\tau = \mu(z)$, where $\mu(z)$ is given by (1.2). The wide neighborhood $\mathcal{N}^-_\infty(\alpha)$, in which we will work in the present paper, is given by

$$\mathcal{N}^-_\infty(\alpha) = \{ z \in \mathcal{F}^0 \ : \ \delta^-_\infty(z) \le \alpha \} ,$$

where $0 < \alpha < 1$ is a given parameter and

$$\delta^-_\infty(z) := \left\| \left[\frac{xs}{\mu(z)} - e \right]^- \right\|_\infty$$

is a proximity measure of z to the central path. Alternatively, if we denote

$$\mathcal{D}(\beta) = \{ z \in \mathcal{F}^0 : xs \ge \beta \mu(z) \},$$

then the neighborhood $\mathcal{N}^-_\infty(\alpha)$ can also be written as

$$\mathcal{N}^-_\infty(\alpha) = \mathcal{D}(1 - \alpha).$$

3 The first order predictor-corrector method

In the predictor step we are given a point $z = \lceil x, \, s \rfloor \in \mathcal{D}(\beta)$, where β is a given parameter in the interval (0, 1), and we compute the affine scaling direction at z:

$$w = \lceil u, \, v \rfloor = -F_0'(z)^{-1} F_0(z) . \tag{3.1}$$

We want to move along that direction as far as possible while preserving the condition $z(\theta) \in \mathcal{D}\left((1 - \gamma)\beta\right)$. The predictor step length is defined as

$$\overline{\theta} = \sup \left\{ \tilde{\theta} > 0 \ : \ z(\theta) \in \mathcal{D}((1 - \gamma)\beta) \ , \forall \, \theta \in [0, \, \tilde{\theta}] \right\} , \tag{3.2}$$

where

$$z(\theta) = z + \theta w$$

and

$$\gamma := \frac{1-\beta}{(1+4\kappa)n+1}\,. \tag{3.3}$$

The output of the predictor step is the point

$$\overline{z} = \lceil\, \overline{x},\, \overline{s}\, \rfloor = z(\overline{\theta}) \in \mathcal{D}((1-\gamma)\beta). \tag{3.4}$$

In the corrector step we are given a point $\overline{z} \in \mathcal{D}((1-\gamma)\beta)$ and we compute the Newton direction of $F_{\mu(\overline{z})}$ at $\overline{z}$:

$$\overline{w} = \lceil\, \overline{u}\,,\, \overline{v}\, \rfloor = -F'_{\mu(\overline{z}\,)}(\overline{z}\,)^{-1}F_{\mu(\overline{z}\,)}(\overline{z}\,)\,, \tag{3.5}$$

which is also known as the centering direction at $\overline{z}$. We denote

$$\overline{x}\,(\theta) = \overline{x} + \theta\overline{u}\,,\ \overline{s}\,(\theta) = \overline{s} + \theta\overline{v}\,,\ \overline{z}\,(\theta) = \lceil\, \overline{x}\,(\theta),\, \overline{s}\,(\theta)\, \rfloor,\ \overline{\mu} = \mu(\overline{z}\,),\ \overline{\mu}\,(\theta) = \mu(\overline{z}\,(\theta)) \tag{3.6}$$

and we determine the corrector step length as

$$\theta_+ = \operatorname{argmin}\,\{\overline{\mu}(\theta) : \overline{z}(\theta) \in \mathcal{D}(\beta)\}. \tag{3.7}$$

The output of the corrector is the point

$$z^+ = \lceil\, x^+,\, s^+\, \rfloor = \overline{z}(\theta_+) \in \mathcal{D}(\beta). \tag{3.8}$$

Since $z^+ \in \mathcal{D}(\beta)$ we can set $z \leftarrow z^+$ and start another predictor-corrector iteration. This leads to the following algorithm.

Algorithm 1
Given $\kappa \geq \chi(Q, R)$, $\beta \in (0,\, 1)$ and $z^0 \in \mathcal{D}(\beta)$:
  Compute γ from (3.3);
  Set $\mu_0 \leftarrow \mu(z^0)$, $k \leftarrow 0$;
  repeat
    (predictor step)
    Set $z \leftarrow z^k$;
    r_1. Compute affine scaling direction (3.1);
    r_2. Compute predictor steplength (3.2);
    r_3. Compute $\overline{z}$ from (3.4);
    If $\mu(\overline{z}\,) = 0$ then STOP: $\overline{z}$ is an optimal solution;
    If $\overline{z} \in \mathcal{D}(\beta)$, then set $z^{k+1} \leftarrow \overline{z}$, $\mu_{k+1} \leftarrow \mu(\overline{z}\,)$, $k \leftarrow k+1$,
      and RETURN;
    (corrector step)
    r_4. Compute centering direction (3.5);
    r_5. Compute centering steplength (3.7);
    r_6. Compute z^+ from (3.8);
  Set $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow \mu(z^+)$, $k \leftarrow k+1$, and RETURN;
  until some stopping criterion is satisfied.

A standard stopping criterion is

$$x^{k\,T}s^k \leq \epsilon\,. \tag{3.9}$$

We will see that if the problem has a solution for any $\epsilon > 0$ Algorithm 1 terminates in a finite number (say K_ϵ) of iterations. If $\epsilon = 0$ then the problem is likely to generate an infinite sequence. However it may happen that at a certain iteration (let us say at iteration K_0) an exact solution is obtained, and therefore the algorithm terminates at iteration K_0. If this (unlikely) phenomenon does not happen we set $K_0 = \infty$.

We now describe some possible implementations of the steps in the **repeat** segment of Algorithm 1:

- In step r_1 of Algorithm 1, the affine scaling direction $w = \lceil u,\, v \rfloor$ can be computed as the solution of the following linear system:

$$\begin{array}{rcl} su + xv & = & -xs \\ Qu + Rv & = & 0\,. \end{array} \tag{3.10}$$

- In step r_2, we find the largest θ that satisfies

$$x(\theta)s(\theta) \geq (1-\gamma)\beta\mu(\theta),$$

where

$$x(\theta) = x + \theta u,\; s(\theta) = s + \theta v,\; \mu = \mu(z),\; \mu(\theta) = \mu(z(\theta)) = x(\theta)^T s(\theta)/n\,. \tag{3.11}$$

According to (3.10) we have

$$x(\theta)s(\theta) = (1-\theta)xs + \theta^2 uv\,, \quad \mu(\theta) = (1-\theta)\mu + \theta^2 u^T v/n\,. \tag{3.12}$$

Let us denote

$$p = \frac{xs}{\mu}\,,\; q = \frac{uv}{\mu}\,.$$

From lemmas 3.1 and 3.2 (to be proved later in this section) it follows that $-\kappa n \leq e^T q \leq .25n$, which implies that the discriminant of the quadratic equation $\mu(\theta) = 0$ is always non-negative. The smallest positive root of $\mu(\theta) = 0$ is

$$\overline{\theta}_0 = \frac{2}{1 + \sqrt{1 - 4e^T q/n}}. \tag{3.13}$$

Therefore

$$\mu(\theta) > \mu(\overline{\theta}_0) = 0\,, \quad \text{for all} \quad 0 \leq \theta < \overline{\theta}_0\,. \tag{3.14}$$

The relation

$$x(\theta)s(\theta) \geq (1-\gamma)\beta\mu(\theta) \tag{3.15}$$

can be written as the following system of quadratic inequalities

$$(1-\theta)\left(p_i - (1-\gamma)\beta\right) + \theta^2\left(q_i - (1-\gamma)\beta e^T q/n\right) \geq 0\,,\; i = 1,\ldots,n\,. \tag{3.16}$$

Since $z \in \mathcal{D}(\beta)$ the above inequalities are satisfied for $\theta = 0$.
The i-th inequality above holds for all $\theta \in (0, \overline{\theta}_i\,]$, where

$$\overline{\theta}_i = \begin{cases} \infty & \text{if } \Delta_i \leq 0 \\ 1 & \text{if } q_i - (1-\gamma)\beta e^T q/n = 0 \\ \frac{2(p_i - (1-\gamma)\beta)}{p_i - (1-\gamma)\beta + \sqrt{\Delta_i}} & \text{if } \Delta_i > 0 \text{ and } q_i - (1-\gamma)\beta e^T q/n \neq 0 \end{cases} \tag{3.17}$$

where

$$\Delta_i = (p_i - (1-\gamma)\beta)^2 - 4\,(p_i - (1-\gamma)\beta)\left(q_i - (1-\gamma)\beta e^T q/n\right) .$$

is the discriminant of the i'th quadratic function in (3.16). By taking

$$\overline{\theta} = \min\left\{\overline{\theta}_i \ : \ 0 \le i \le n\right\} . \tag{3.18}$$

we have

$$x(\theta)s(\theta) \ge (1-\gamma)\beta\mu(\theta) > (1-\gamma)\beta\mu(\overline{\theta}) \ge 0\,, \quad \text{for all} \quad 0 \le \theta < \overline{\theta}\,. \tag{3.19}$$

From (3.10) and (3.11) it follows that $Qx(\theta) + Rs(\theta) = b$, and by using a standard continuity argument we can prove that $x(\theta) > 0$, $s(\theta) > 0$ for all $\theta \in (0, \overline{\theta}\,)$, which implies that the point $\overline{z} = z(\overline{\theta}\,)$ given by the predictor step satisfies $\overline{z} \in \mathcal{F}^0$.
If $\overline{\theta} = \overline{\theta}_0$, then $\mu(\overline{z}\,) = 0$, so that $\overline{z}$ is an optimal solution of our problem, i.e. $\overline{z} \in \mathcal{F}^*$; If $\mu(\overline{z}\,) > 0$, then $\overline{z} \in \mathcal{D}((1-\gamma)\beta)$. If $\overline{z} \notin \mathcal{D}(\beta)$, a corrector step is performed.

- In step r_4 of Algorithm 1, the centering direction can be computed as the solution of the following linear system

$$\begin{array}{rcl} \overline{s}\,\overline{u} + \overline{x}\,\overline{v} & = & \mu(\overline{z}\,) - \overline{x}\,\overline{s} \\ Q\overline{u} + R\overline{v} & = & 0\,. \end{array} \tag{3.20}$$

- In step r_5, we determine the corrector step length θ_+ as follows:
From (3.6) and (3.20) it follows that

$$\overline{x}\,(\theta)\overline{s}\,(\theta) = (1-\theta)\overline{x}\,\overline{s} + \theta\overline{\mu} + \theta^2\overline{u}\,\overline{v}\;, \quad \overline{\mu}\,(\theta) = \overline{\mu} + \theta^2\overline{u}^{\,T}\overline{v}\,/n\,. \tag{3.21}$$

The relation

$$\overline{x}\,(\theta)\overline{s}\,(\theta) \ge \beta\overline{\mu}\,(\theta) \tag{3.22}$$

is equivalent to the following system of quadratic inequalities in θ

$$\overline{f}_{\,i}(\theta) := \overline{p}_{\,i} - \beta + \theta\,(1 - \overline{p}_{\,i}) + \theta^2\left(\overline{q}_{\,i} - \beta e^T\overline{q}\,/n\right) \ge 0\,,\ i = 1,\ldots,n \tag{3.23}$$

where

$$\overline{p} = \frac{\overline{x}\,\overline{s}}{\overline{\mu}}\,,\, \overline{q} = \frac{\overline{u}\,\overline{v}}{\overline{\mu}}\,.$$

Let us denote the leading coefficient by $\overline{\alpha}_i$ and the discriminant of $\overline{f}_{\,i}(\theta)$ by $\overline{\Delta}_i$:

$$\overline{\alpha}_i = \overline{q}_{\,i} - \beta e^T\overline{q}\,/n\,, \quad \overline{\Delta}_i = (1-\overline{p}_{\,i})^2 - 4\left(\overline{q}_{\,i} - \beta e^T\overline{q}\,/n\right)(\overline{p}_{\,i} - \beta)\ .$$

If $\overline{\Delta}_i \ge 0$ and $\overline{\alpha}_i \ne 0$ we denote by $\check{\overline{\theta}}_{\,i}$ and $\hat{\overline{\theta}}_{\,i}$ the smallest and the largest root of $\overline{f}_{\,i}(\theta)$ respectively, i.e.

$$\check{\overline{\theta}}_{\,i} = \frac{\overline{p}_{\,i} - 1 - \operatorname{sign}(\overline{\alpha}_i)\sqrt{\overline{\Delta}_i}}{2\overline{\alpha}_i}, \quad \hat{\overline{\theta}}_{\,i} = \frac{\overline{p}_{\,i} - 1 + \operatorname{sign}(\overline{\alpha}_i)\sqrt{\overline{\Delta}_i}}{2\overline{\alpha}_i}\,.$$

In the proof of Theorem 3.3 we will show that (3.23) has a solution, so that the following situation cannot occur for any $i = 1,\ldots,n$

$$\overline{\Delta}_i < 0 \text{ and } \overline{\alpha}_i < 0\,.$$

By analyzing all possible situations, we conclude that the i'th inequality in (3.23) will be satisfied for all $\theta \in \mathcal{T}_i$, where

$$\mathcal{T}_i = \begin{cases} (-\infty\,,\,\infty)\,, & \text{if } \overline{\Delta}_i < 0 \text{ and } \overline{\alpha}_i > 0 \\ (-\infty,\,\check{\overline{\theta}}_i] \cup [\hat{\overline{\theta}}_i\,,\,\infty), & \text{if } \overline{\Delta}_i \geq 0 \text{ and } \overline{\alpha}_i > 0 \\ \left[\check{\overline{\theta}}_i\,,\,\hat{\overline{\theta}}_i\right], & \text{if } \overline{\Delta}_i \geq 0 \text{ and } \overline{\alpha}_i < 0 \\ (-\infty\,,\,(\overline{p}_i - \beta)/(\overline{p}_i - 1)\,]\,, & \text{if } \overline{\alpha}_i = 0 \text{ and } \overline{p}_i > 1 \\ [\,(\overline{p}_i - \beta)/(\overline{p}_i - 1)\,,\,\infty)\,, & \text{if } \overline{\alpha}_i = 0 \text{ and } \overline{p}_i < 1 \\ (-\infty\,,\,\infty)\,, & \text{if } \overline{\alpha}_i = 0 \text{ and } \overline{p}_i = 1 \end{cases} .$$

It follows that (3.22) holds for all $\theta \in \mathcal{T}$ where

$$\mathcal{T} = \bigcap_{i=1}^{n} \mathcal{T}_i \bigcap R^n_+. \tag{3.24}$$

We define the step length θ_+ for the corrector by

$$\theta_+ = \begin{cases} \min_{\theta \in \mathcal{T}} \theta & if \quad u^T v \geq 0 \\ \max_{\theta \in \mathcal{T}} \theta & if \quad u^T v < 0 \end{cases} . \tag{3.25}$$

It can be proved that $\mathcal{T}$ is bounded below when $u^T v \geq 0$ and is bounded above when $u^T v < 0$. In proof of Theorem 3.3 we will show that $\mathcal{T}$ is non-empty so that (3.25) is well defined. We notice by (3.25) and (3.11),

$$\overline{\mu}\,(\theta_+) \leq \overline{\mu}\,(\theta)\,, \quad \forall \theta \in \mathcal{T}\,. \tag{3.26}$$

With θ_+ determined as above, the corrector step produces a point and another predictor-corrector iteration can be performed.

Polynomial Complexity. In what follows we will prove that the step length $\overline{\theta}$ computed in the predictor step is bounded below by a quantity of the form $\sigma/((1+4\kappa)n+2)$, where σ is a positive constant. This implies that Algorithm 1 has $O((1+\kappa)nL)$-iteration complexity.

The following two technical lemmas will be used in the proof of our main result.

LEMMA 3.1. *Assume that HLCP (2.1) is $P_*(\kappa)$, and let $w = \lceil\, u,\, v\, \rfloor$ be the solution of the following linear system*

$$\begin{aligned} su + xv &= a \\ Qu + Rv &= 0 \end{aligned}$$

where $z = \lceil\, x,\, s\, \rfloor \in I\!R^{2n}_{++}$ and $a \in I\!R^n$ are given vectors, and consider the index sets:

$$\mathcal{I}^+ = \{i : u_i v_i > 0\}, \quad \mathcal{I}^- = \{i : u_i v_i < 0\}.$$

Then the following inequalities are satisfied:

$$\frac{1}{1+4\kappa}\,\|\,u\,v\,\|_\infty \le \sum_{i\in\mathcal{I}_+} u_i v_i \le \frac{1}{4}\left\|\,(xs)^{-1/2}\,a\,\right\|_2^2 .$$

Proof The second inequality is well known for the monotone HLCP and it extends trivially to the $P_*(\kappa)$ HLCP. To prove the first inequality, we assume that for index t we have

$$|u_t v_t| = \max_i |u_i v_i| = \|\,u\,v\,\|_\infty .$$

If $u_t v_t \ge 0$, then

$$|u_t v_t| = u_t v_t \le \sum_{i\in\mathcal{I}^+} u_i v_i;$$

If $u_t v_t < 0$, then

$$|u_t v_t| = -u_t v_t \le -\sum_{i\in\mathcal{I}^-} u_i v_i \le (1+4\kappa)\sum_{i\in\mathcal{I}^+} u_i v_i;$$

Thus the first inequality holds in either case.

■

LEMMA 3.2. *Assume that HLCP (2.1) is $P_*(\kappa)$, and let $w = \lceil\, u,\, v\,\rfloor$ be the solution of the following linear system*

$$\begin{aligned} su + xv &= a \\ Qu + Rv &= 0 \end{aligned}$$

where $z = \lceil\, x,\, s\,\rfloor \in I\!R^{2n}_{++}$ and $a \in I\!R^n$ are given vectors. Then the following inequality holds:

$$u^T v \ge -\kappa \left\|\,(xs)^{-1/2}\,a\,\right\|_2^2 . \tag{3.27}$$

Proof Using Lemma 3.1 we can write

$$\begin{aligned} u^T v &= \sum_{i\in\mathcal{I}^+} u_i v_i + \sum_{i\in\mathcal{I}^-} u_i v_i = (1+4\kappa)\sum_{i\in\mathcal{I}^+} u_i v_i + \sum_{i\in\mathcal{I}^-} u_i v_i - 4\kappa\sum_{i\in\mathcal{I}^+} u_i v_i \\ &\ge -4\kappa\sum_{i\in\mathcal{I}^+} u_i v_i \ge -\kappa\left\|\,(xs)^{-1/2}\,a\,\right\|_2^2 . \end{aligned}$$

■

The following result implies that Algorithm 1 has $O((1+\kappa)nL)$ iteration complexity.

THEOREM 3.3. *Algorithm 1 is well defined and*

$$\mu_{k+1} \le \left(1 - \frac{3\sqrt{(1-\beta)\beta}}{2((1+4\kappa)n+2)}\right)\mu_k, \quad k = 0, 1, \ldots \tag{3.28}$$

Proof According to Lemma 3.1 and Lemma 3.2 we have

$$\| q \|_\infty \le .25(1+4\kappa)n, \quad -\kappa n \le e^T q \le \sum_{i\in\mathcal{I}_+(w)} q_i \le .25n\,. \tag{3.29}$$

Moreover, in the predictor step we have $z \in \mathcal{D}(\beta)$, so that $p_i - (1-\gamma)\beta > 0$. Hence the quantity defined in (3.17) satisfies

$$\overline{\theta}_i \ge \frac{2(p_i - (1-\gamma)\beta)}{p_i - (1-\gamma)\beta + \sqrt{(p_i-(1-\gamma)\beta)^2 - 4(p_i-(1-\gamma)\beta)(q_i - (1-\gamma)\beta\frac{e^Tq}{n})}}$$

$$\ge \frac{2(p_i - (1-\gamma)\beta)}{p_i - (1-\gamma)\beta + \sqrt{(p_i-(1-\gamma)\beta)^2 + 4(p_i-(1-\gamma)\beta)(\|q\|_\infty + \frac{1}{4}(1-\gamma)\beta)}}.$$

Since the function $t \to t/(t+\sqrt{t^2+4at})$ is increasing on $(0,\infty)$ for any $a>0$, we deduce from above that

$$\overline{\theta}_i \ge \frac{2\beta\gamma}{\beta\gamma + \sqrt{\beta^2\gamma^2 + 4\beta\gamma(\| q \|_\infty + 1/4)}} = \frac{2}{1+\sqrt{1+(\beta\gamma)^{-1}(4\| q \|_\infty + 1)}}$$

$$\ge \frac{2}{1+\sqrt{1+(\beta\gamma)^{-1}((1+4\kappa)n+1)}} = \frac{2\sqrt{\beta(1-\beta)}}{\sqrt{\beta(1-\beta)} + \sqrt{((1+4\kappa)n+1)^2 + \beta(1-\beta)}}\,.$$

Since

$$\sqrt{\beta(1-\beta)} + \sqrt{((1+4\kappa)n+1)^2+\beta(1-\beta)}$$
$$\le .5 + \sqrt{((1+4\kappa)n+1)^2 + .25} < (1+4\kappa)n+2$$

we deduce that

$$\overline{\theta_i} > \tilde{\theta} := \frac{2\sqrt{(1-\beta)\beta}}{(1+4\kappa)n+2}\,. \tag{3.30}$$

Notice that for any $\kappa > 0$ and $n \ge 1$,

$$\overline{\theta}_0 \ge \frac{2}{1+\sqrt{1+4\kappa}} > \tilde{\theta}.$$

It follows that the quantity defined in (3.18) satisfies $\overline{\theta} > \tilde{\theta}$. Relations (3.19), (3.12) and (3.29) imply

$$\overline{\mu} = \mu(\overline{\theta}\,) < \mu(\tilde{\theta}) \le \Big((1-\tilde{\theta}) + .25\,\tilde{\theta}^{\,2}\Big)\,\mu = \Big(1 - (1-.25\,\tilde{\theta}\,)\,\tilde{\theta}\Big)\,\mu\,.$$

Since we assume that $n \ge 2$ and $\kappa \ge 0$, we have

$$1 - .25\,\tilde{\theta} = 1 - \frac{\sqrt{(1-\beta)\beta}}{2((1+4\kappa)n+2)} \ge 1 - \frac{\sqrt{(1-\beta)\beta}}{8} \ge 1 - \frac{1}{16} = \frac{15}{16}$$

and we obtain

$$\overline{\mu} \le \left(1 - \frac{15\sqrt{(1-\beta)\beta}}{8((1+4\kappa)n+2)}\right)\mu\,. \tag{3.31}$$

Let us analyze now the corrector. Since $\overline{z} \in \mathcal{D}((1-\gamma)\beta)$, we have

$$\left\| \sqrt{\frac{\overline{x}\,\overline{s}}{\overline{\mu}}} - \sqrt{\frac{\overline{\mu}}{\overline{x}\,\overline{s}}} \right\|_2^2 = \sum_{i=1}^n \frac{\overline{\mu}}{\overline{x}_i \overline{s}_i} - 2n + \sum_{i=1}^n \frac{\overline{x}_i \overline{s}_i}{\overline{\mu}} = \sum_{i=1}^n \frac{\overline{\mu}}{\overline{x}_i \overline{s}_i} - n \le \frac{1-(1-\gamma)\beta}{(1-\gamma)\beta}\, n$$

and by applying Lemma 3.1 we deduce that

$$\|\,\overline{u}\,\overline{v}\,\|_\infty \le \frac{(1+4\kappa)\xi n}{4}\,\overline{\mu}\ , \qquad \sum_{i\in\mathcal{I}_+(\overline{w})} \overline{u}_i \overline{v}_i \le \frac{\xi n}{4}\,\overline{\mu}\ ,\ \text{ where } \xi := \frac{1-(1-\gamma)\beta}{(1-\gamma)\beta}\,. \tag{3.32}$$

By substituting γ into the above equation, we get

$$\xi = \frac{(1-\beta)((1+4\kappa)n+1+\beta)}{((1+4\kappa)n+\beta)\beta}\,. \tag{3.33}$$

From (3.21) it follows that

$$\begin{aligned}
\frac{\overline{x}(\theta)\overline{s}(\theta)}{\overline{\mu}} &\ge (1-\theta)(1-\gamma)\beta + \theta - \frac{(1+4\kappa)n}{4}\xi\theta^2 \\
&= (1-\gamma)\beta + \theta(1-(1-\gamma)\beta) - \frac{(1+4\kappa)n}{4}\xi\theta^2
\end{aligned}$$

and

$$\overline{\mu}(\theta) \le \left(1 + .25\,\xi\theta^2\right)\overline{\mu}\ . \tag{3.34}$$

Therefore

$$\frac{\overline{x}(\theta)\overline{s}(\theta) - \beta\overline{\mu}(\theta)}{\overline{\mu}} \ge g(\theta) := -\gamma\beta + \theta(1-(1-\gamma)\beta) - .25\xi((1+4\kappa)n+\beta)\theta^2\,.$$

Using (3.33) together with the definition (3.3) of γ we obtain

$$\begin{aligned}
g(\theta) \;=\; & -\frac{1-\beta}{4\beta((1+4\kappa)n+1)} \\
& \big((((1+4\kappa)n+1)\theta - 2\beta)(((1+4\kappa)n+1+\beta)\theta - 2\beta) - 2\beta^2\theta\big)\,.
\end{aligned}$$

Since

$$g\left(\frac{2\beta}{(1+4\kappa)n+1}\right) = \frac{(1-\beta)\beta^2}{((1+4\kappa)n+1)^2} \ge 0\,,$$

we deduce that $2\beta/((1+4\kappa)n+1) \in \mathcal{T}$. By using (3.34) and $n \ge 2$ we have

$$\mu_+ = \overline{\mu}(\theta_+) \le \overline{\mu}\left(\frac{2\beta}{(1+4\kappa)n+1}\right) \le \left(1 + \frac{(1-\beta)\beta((1+4\kappa)n+1+\beta)}{((1+4\kappa)n+1)^2((1+4\kappa)n+\beta)}\right)\overline{\mu}\ .$$

Given that $n \ge 2$ and $\kappa \ge 0$, we have $\frac{(1+4\kappa)n+1+\beta}{(1+4\kappa)n+\beta} < \frac{3}{2}$, so that the above inequality implies

$$\mu_+ \le \left(1 + \frac{3(1-\beta)\beta}{2((1+4\kappa)n+1)^2}\right)\overline{\mu}\ . \tag{3.35}$$

Finally, by using (3.31), we obtain

$$\begin{aligned}
\mu_+ &\le \left(1-\frac{15\sqrt{(1-\beta)\beta}}{8((1+4\kappa)n+2)}\right)\left(1+\frac{3(1-\beta)\beta}{2((1+4\kappa)n+1)^2}\right)\mu \\
&\le \left(1-\frac{15\sqrt{(1-\beta)\beta}}{8((1+4\kappa)n+2)}\right)\left(1+\frac{3\beta\,(1-\beta)}{2(1+4\kappa)n((1+4\kappa)n+2)}\right)\mu \\
&\le \left(1-\frac{15\sqrt{(1-\beta)\beta}}{8((1+4\kappa)n+2)}+\frac{3\beta\,(1-\beta)}{2(1+4\kappa)n((1+4\kappa)n+2)}\right)\mu \\
&\le \left(1-\left(\frac{15}{8}-\frac{3\sqrt{(1-\beta)\beta}}{2(1+4\kappa)n}\right)\frac{\sqrt{(1-\beta)\beta}}{((1+4\kappa)n+2)}\right)\mu \\
&\le \left(1-\frac{3\sqrt{(1-\beta)\beta}}{2((1+4\kappa)n+2)}\right)\mu\,.
\end{aligned}$$

The last inequality holds since $\sqrt{(1-\beta)\beta} \le .5$ and $(1+4\kappa)n \ge 2$ imply

$$\frac{15}{8}-\frac{3\sqrt{(1-\beta)\beta}}{2(1+4\kappa)n} \ge \frac{3}{2}. \tag{3.36}$$

The proof is complete. ■

The next corollary is an immediate consequence of the above theorem.

COROLLARY 3.4. *Algorithm 1 with stopping criterion (3.9) produces a point* $z^k \in \mathcal{D}(\beta)$ *with* ${x^k}^T s^k \le \varepsilon$ *in at most* $O\left((1+\kappa)n\log\left({x^0}^T s^0/\varepsilon\right)\right)$ *iterations.*

Quadratic Convergence We will next prove that the sequence $\{\mu(z^k)\}$ is quadratically convergent to zero in the sense that

$$\mu_{k+1} = O(\mu_k^2)\,. \tag{3.37}$$

We do need the assumption that our $P_*(\kappa)$ HLCP is nondegenerate, i.e. the set of all strictly complementary solutions

$$\mathcal{F}^{\#} := \{z^* = \lceil x^*,\, s^* \rfloor \in \mathcal{F}^* \;:\; x^* + s^* > 0\}$$

is non-empty. In fact, this assumption is not restrictive and it is needed for a large class of interior-point methods using only first order derivatives to obtain superlinear convergence, including the MTY predictor-corrector method [4, 13].

The following result was proved for standard monotone LCP in [26] and for HLCP in [4]. Its extension for $P_*(\kappa)$ HLCP is straightforward (see [2]).

LEMMA 3.5. *If* $\mathcal{F}^{\#} \ne \emptyset$ *then the solution* $w = \lceil u,\, v \rfloor$ *of (3.10) satisfies*

$$|\, u_i v_i \,| = O(\mu^2),\ \forall i \in \{1, 2, \ldots, n\}\,,$$

where $\mu = \mu(z)$ *is given by (1.2).*

With the help of this lemma the quadratic convergence result of [15] automatically extends to our case.

THEOREM 3.6. *If HLCP has a strictly complementary solution, then the sequence* $\{\mu_k\}$ *generated by Algorithm 1 with no stopping criterion converges quadratically to zero in the sense that (3.37) is satisfied.*

Algorithm 1 depends on a given parameter $\kappa \geq \chi(Q,R)$ because of the choice of γ from (3.3). However, in many applications it may very expensive to find a good upper bound for the handicap $\chi(Q,R)$ [23]. Therefore we propose an algorithm that does not depend on κ. Initially we set $\kappa = 1$ and use Algorithm 1 for this value of κ. If at a certain iteration the corrector fails to produce a point in $\mathcal{D}(\beta)$, then we conclude that the current value of κ is too small. In this case we double the value of κ and restart Algorithm 1 from the last point produced in $\mathcal{D}(\beta)$. Clearly we have to double the value of κ at most $\lceil \log_2 \chi(Q,R) \rceil$ times. This leads to the following algorithm.

Algorithm 1A

Given $\beta \in (0,\, 1)$ and $z^0 \in \mathcal{D}(\beta)$:

    Set $\kappa \leftarrow 1$, $\mu_0 \leftarrow \mu(z^0)$, $k \leftarrow 0$;
    repeat
        Compute γ from (3.3);
        (predictor step)
        Set $z \leftarrow z^k$;
        r_1. Compute affine scaling direction (3.1);
        r_2. Compute predictor steplength (3.2);
        r_3. Compute $\overline{z}$ from (3.4);
        If $\mu(\overline{z}) = 0$ then STOP: $\overline{z}$ is an optimal solution;
        If $\overline{z} \in \mathcal{D}(\beta)$, then set $z^{k+1} \leftarrow \overline{z}$, $\mu_{k+1} \leftarrow \mu(\overline{z})$, $k \leftarrow k+1$,
            and RETURN;
        (corrector step)
        r_4. Compute centering direction (3.5);
        r_5. Compute centering steplength (3.7);
        r_6. Compute z^+ from (3.8);
        if $z^+ \in \mathcal{D}(\beta)$, set $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow \mu(z^+)$, $k \leftarrow k+1$,
            and RETURN;
        else, set $\kappa \leftarrow 2\kappa$ and $z^{k+1} \leftarrow z^k$, $\mu_{k+1} \leftarrow \mu(z^k)$, $k \leftarrow k+1$,
            and RETURN;
    until some stopping criterion is satisfied.

Using Theorem 3.3, Corollary 3.4, and, Theorem 3.6 we obtain the following result.

THEOREM 3.7. *Algorithm 1A with stopping criterion (3.9) produces a point* $z^k \in \mathcal{D}(\beta)$ *with* $x^{k\,T} s^k \leq \varepsilon$ *in at most* $O\left((1+\chi(Q,R))\, n \log\left(x^{0\,T} s^0/\varepsilon\right)\right)$ *iterations. If HLCP has a strictly complementary solution, then the sequence* $\{\mu_k\}$ *generated by Algorithm 1A with no stopping criterion converges quadratically to zero in the sense that (3.37) is satisfied.*

Proof Let $\overline{\kappa}$ be the largest value of κ used in Algorithm 1A. We have clearly $\overline{\kappa} < 2\chi(Q,R)$. Consider now that at iteration k of Algorithm 1A we have $\kappa < \chi(Q,R)$. If the corrector step is accepted, i.e. if $z^+ \in \mathcal{D}(\beta)$, then $z^{k+1} = z^+$ and by inspecting the proof of (3.28) it follows that

$$\mu_{k+1} \leq \left(1 - \frac{3\sqrt{(1-\beta)\beta}}{2((1+4\chi(Q,R))n+2)}\right)\mu_k \leq \left(1 - \frac{3\sqrt{(1-\beta)\beta}}{2((1+8\chi(Q,R))n+2)}\right)\mu_k\,.$$

This is easily seen because Lemma 3.1 and Lemma 3.2 hold for $\kappa = \chi(Q,R)$ while the bound on the predictor step size depends on γ which is decreasing in κ.

If $\kappa \geq \chi(Q,R)$ then the corrector step is never rejected and from (3.28) and the fact $\kappa \leq \overline{\kappa} < 2\chi(Q,R)$, we have

$$\mu_{k+1} \leq \left(1 - \frac{3\sqrt{(1-\beta)\beta}}{2((1+4\kappa)n+2)}\right)\mu_k \leq \left(1 - \frac{3\sqrt{(1-\beta)\beta}}{2((1+8\chi(Q,R))n+2)}\right)\mu_k\,.$$

Since there can be at most $\log_2 \overline{\kappa}$ rejections we obtain the desired complexity result. In case the problem is nondegenerate we always have $\mu_+ = O(\mu^2)$ and since there are only a finite number of corrector rejections it follows that for sufficiently large k we have $\mu_{k+1} = O(\mu_k^2)$. ∎

Let us end this section by remarking that even if $\chi(Q,R)$ is known, it is not very clear whether Algorithm 1 with $\kappa = \chi(Q,R)$ is more efficient than Algorithm 1A on a particular problem. Indeed it may happen that the corrector step in Algorithm 1A is accepted for smaller values of κ for some iterations, and those iterations will yield a better reduction of the complementarity gap.

4 A higher order predictor corrector

The higher order predictor uses higher derivatives of the central path. Given a point $z = \lceil x,\, s \rfloor \in \mathcal{D}(\beta)$ we consider the curve given by

$$z(\theta) = z + \sum_{i=1}^{m} w^i \theta^i \tag{4.1}$$

where w^1 is just the affine scaling direction used in the first order predictor, and w^i are the directions related to the higher derivatives of the central path (see [20]). The vectors $w^i = \lceil u^i,\, v^i \rfloor$ can be obtained as the solutions of the following linear systems

$$\begin{aligned}
&\begin{cases} su^1 + xv^1 &= \gamma\mu e - (1+\epsilon)xs \\ Qu^1 + Rv^1 &= 0 \end{cases}, \\
&\begin{cases} su^2 + xv^2 &= \epsilon xs - u^1 v^1 \\ Qu^2 + Rv^2 &= 0 \end{cases}, \\
&\begin{cases} su^i + xv^i &= -\sum_{j=1}^{i-1} u^j v^{i-j} \\ Qu^i + Rv^i &= 0 \end{cases}, \quad i = 3,\ldots,m\,,
\end{aligned} \tag{4.2}$$

where

$$\epsilon = \begin{cases} 0\,, & \text{if HLCP is nondegenerate} \\ 1\,, & \text{if HLCP is degenerate} \end{cases} . \tag{4.3}$$

The m linear systems above have the same matrix, so that their numerical solution requires only one matrix factorization and m backsolves. This involves $O(n^3) + m\,O(n^2)$ arithmetic operations.

Since the case $m = 1$ has been analyzed in the previous section, for the remainder of this paper we assume that $m \geq 2$. Given predictor (4.1), we want to choose the step size $\overline{\theta}$ such that we have $\mu(\theta)$ as small as possible while still keeping the point in the neighborhood $\mathcal{D}((1-\gamma)\beta)$.

We define

$$\breve{\theta} = \sup\left\{\tilde{\theta} > 0 \;:\; z(\theta) \in \mathcal{D}((1-\gamma)\beta)\,, \forall\, \theta \in [0, \tilde{\theta}]\right\}\,, \tag{4.4}$$

where γ is given by (3.3), and β is a given parameter chosen in the interval (0,1). From (4.1) - (4.2) we deduce that

$$\begin{aligned} x(\theta)s(\theta) &= (1-\theta)^{1+\epsilon}xs + \sum_{i=m+1}^{2m} \theta^i h^i\,, \\ \mu(\theta) &= (1-\theta)^{1+\epsilon}\mu + \sum_{i=m+1}^{2m} \theta^i (e^T h^i/n)\,, \\ &\qquad \text{where } h^i = \sum_{j=i-m}^{m} u^j v^{i-j}\,. \end{aligned} \tag{4.5}$$

Therefore the computation of (4.4) involves the solution of a system of polynomial inequalities of order $2m$ in θ. While is possible to obtain an accurate lower bound of the exact solution by using linear search, in the present paper we only give a simple lower bound in explicit form which is sufficiently good for proving our theoretical results. The predictor step length $\overline{\theta}$ is chosen to minimize $\mu(\theta)$ in the interval $[\,0, \breve{\theta}\,]$, i.e.

$$\overline{\theta} = \text{argmin}\left\{\mu(\theta) \;:\; \theta \in [\,0, \breve{\theta}\,]\right\}\,, \quad \overline{z} = z(\overline{\theta}\,)\,. \tag{4.6}$$

We have $\overline{z} \in \mathcal{D}((1-\gamma)\beta)$ by construction. Using the same corrector as in the previous section we obtain $z^+ \in \mathcal{D}(\beta)$. By replacing the predictor in Algorithm 1, with the predictor described above we obtain:

Algorithm 2

Given $\kappa \geq \chi(Q,R)$, $\beta \in (0,1)$, an integer $m \geq 2$, and $z^0 \in \mathcal{D}(\beta)$:

Compute γ from (3.3);

Set $\mu_0 \leftarrow \mu(z_0)$, $k \leftarrow 0$;

repeat

(predictor step)

Compute directions $w^i = \lceil\, u^i,\, v^i\,\rfloor$, $\quad i = 1, \ldots, m$ by solving (4.2);

Compute $\breve{\theta}$ from (4.4);

Compute $\overline{z}$ from (4.6) ;

If $\mu(\overline{z}\,) = 0$ then STOP: $\overline{z}$ is an optimal solution;

If $\overline{z} \in \mathcal{D}(\beta)$, then set $z^{k+1} \leftarrow \overline{z}$, $\mu_{k+1} \leftarrow \mu(\overline{z})$, $k \leftarrow k+1$,
and RETURN;
(corrector step)
Compute centering direction (3.5) by solving (3.20);
Compute centering steplength (3.25);
Compute z^+ from (3.8);
Set $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow \mu(z^+)$, $k \leftarrow k+1$, and RETURN;
until some stopping criterion is satisfied.

Let us denote

$$\eta_i = \left\| Du^i + D^{-1}v^i \right\|_2 , \text{ where } D = X^{-1/2}S^{1/2} .$$

The following lemma that will be used in the proof of the main result of this section.

LEMMA 4.1. *The solution of (4.2) satisfies*

$$\frac{1}{\sqrt{1+2\kappa}}\sqrt{\| Du^i \|_2^2 + \| D^{-1}v^i \|_2^2} \le \eta_i \le \frac{2}{1+2\kappa}\alpha_i\sqrt{\beta\mu}\left(\frac{(1+2\kappa)(1+\epsilon)}{2}\sqrt{n/\beta}\right)^i ,$$

where the sequence

$$\alpha_i = \frac{1}{i}\left(\begin{array}{c} 2i-2 \\ i-1 \end{array}\right) \le \frac{1}{i}4^i$$

is the solution of the following recurrence scheme

$$\alpha_1 = 1, \quad \alpha_i = \sum_{j=1}^{i-1}\alpha_j\,\alpha_{i-j} .$$

Proof The first part of the inequality follows immediately, since by using (4.2) and Lemma 3.2 we have

$$\begin{aligned}\left\| Du^i + D^{-1}v^i \right\|_2^2 &= \left\| Du^i \right\|_2^2 + 2{u^i}^T v^i + \left\| D^{-1}v^i \right\|_2^2 \\ &\ge \left\| Du^i \right\|_2^2 + \left\| D^{-1}v^i \right\|_2^2 - 2\kappa \left\| Du^i + D^{-1}v^i \right\|_2^2 .\end{aligned}$$

By multiplying the first equations of (4.2) with $(xs)^{-1/2}$ we obtain

$$\begin{aligned} Du^1 + D^{-1}v^1 &= -(1+\epsilon)(xs)^{1/2} \\ Du^2 + D^{-1}v^2 &= \epsilon(xs)^{1/2} - (xs)^{-1/2}u^1v^1 \\ Du^i + D^{-1}v^i &= -(xs)^{-1/2}\sum_{j=1}^{i-1} Du^jD^{-1}v^{i-j} , \quad i = 3,\ldots,m . \end{aligned}$$

Using Lemma3.2, Corollary 2.3 of [14], and the fact $z \in \mathcal{D}(\beta)$, we deduce that

$$\begin{aligned} \eta_1 &= (1+\epsilon)\sqrt{n\mu} , \\ \eta_2^2 &\le \epsilon^2 n\mu - 2\epsilon(u^1)^T(v^1) + \frac{\|u^1v^1\|_2^2}{\beta\mu} \end{aligned}$$

$$\begin{aligned}
&\leq \epsilon^2 n\mu + 2\epsilon\kappa\eta_1^2 + \frac{1}{8\beta\mu}(1+4\kappa+8\kappa^2)\eta_1^4 \\
&= \epsilon^2 n\mu + 2\epsilon\kappa(1+\epsilon)^2 n\mu + \frac{1}{8\beta\mu}(1+4\kappa+8\kappa^2)(1+\epsilon)^4 n^2\mu^2 \,.
\end{aligned}$$

We want to prove that the inequality holds for $i = 2$, i.e.,

$$\eta_2^2 \leq \frac{(1+2\kappa)^2}{4\beta} n^2 \mu (1+\epsilon)^4 \,.$$

This inequality holds provided

$$\epsilon^2 n\mu + 2\epsilon\kappa(1+\epsilon)^2 n\mu \leq \frac{n^2\mu}{8\beta}(1+4\kappa)(1+\epsilon)^4 \,,$$

which is trivially satisfied for both $\epsilon = 0$ and $\epsilon = 1$.
Finally, for $i \geq 3$, we have

$$\eta_i \leq \frac{1}{\sqrt{\beta\mu}} \sum_{j=1}^{i-1} \left\| Du^j \right\|_2 \left\| D^{-1}v^{i-j} \right\|_2 \,, \quad i = 3, \ldots, m \,.$$

Since

$$\begin{aligned}
&\left\| Du^j \right\|_2 \left\| D^{-1}v^{i-j} \right\|_2 + \left\| Du^{i-j} \right\|_2 \left\| D^{-1}v^j \right\|_2 \\
\leq\ & \left(\left\| Du^j \right\|_2^2 + \left\| D^{-1}v^j \right\|_2^2\right)^{1/2} \left(\left\| Du^{i-j} \right\|_2^2 + \left\| D^{-1}v^{i-j} \right\|_2^2\right)^{1/2} \\
\leq\ & (1+2\kappa)\eta_j\, \eta_{i-j} \,,
\end{aligned}$$

we obtain

$$\eta_i \leq \frac{1+2\kappa}{2\sqrt{\beta\mu}} \sum_{j=1}^{i-1} \eta_j\, \eta_{i-j} \,, \quad i = 2, \ldots, m \,.$$

The required inequalities are then easily proved by mathematical induction. ■

THEOREM 4.2. *Algorithm 2 is well defined and for each $n \geq 14$ we have*

$$\mu_{k+1} \leq \left(1 - .0011 \frac{\sqrt{\beta}\,\sqrt[3]{1-\beta}}{(1+2\kappa)^{1+1/(m(m+1))}\sqrt{n}\,\sqrt[m+1]{(1+4\kappa)n+2}}\right)\mu_k \,, \quad k = 0, 1, \ldots$$

Proof An upper bound of $\left\| h^i \right\|_2$, $i = m+1, m+2, \ldots, 2m$ can be obtained by writing

$$\begin{aligned}
\left\| h^i \right\|_2 &\leq \sum_{j=i-m}^{m} \left\| Du^j \right\|_2 \left\| D^{-1}v^{i-j} \right\|_2 \leq \sum_{j=1}^{i-1} \left\| Du^j \right\|_2 \left\| D^{-1}v^{i-j} \right\|_2 \\
&= \frac{1}{2}\sum_{j=1}^{i-1} \left(\left\| Du^j \right\|_2 \left\| D^{-1}v^{i-j} \right\|_2 + \left\| Du^{i-j} \right\|_2 \left\| D^{-1}v^j \right\|_2\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\sum_{j=1}^{i-1}\sqrt{\| Du^j \|_2^2 + \| D^{-1}v^j \|_2^2}\sqrt{\| Du^{i-j} \|_2^2 + \| D^{-1}v^{i-j} \|_2^2} \\
&\leq \frac{1+2\kappa}{2}\sum_{j=1}^{i-1}\eta_j\,\eta_{i-j} \leq \frac{2\beta\mu}{1+2\kappa}(\frac{(1+2\kappa)(1+\epsilon)}{2}\sqrt{n/\beta})^i\sum_{j=1}^{i-1}\alpha_j\,\alpha_{i-j} \\
&= \frac{2\beta\mu}{1+2\kappa}(\frac{(1+2\kappa)(1+\epsilon)}{2}\sqrt{n/\beta})^i\alpha_i \\
&\leq \frac{2\beta\mu}{(1+2\kappa)i}\left(2(1+2\kappa)(1+\epsilon)\sqrt{n/\beta}\right)^i .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left\| \sum_{i=m+1}^{2m}\theta^i h^i \right\|_2 &\leq \frac{2\beta\mu}{(1+2\kappa)(m+1)}\left(4(1+2\kappa)\theta\sqrt{n/\beta}\right)^{m+1}\sum_{j=0}^{m-1}\left(4(1+2\kappa)\theta\sqrt{n/\beta}\right)^j \\
&\leq \frac{2\beta\mu\left(4(1+2\kappa)\theta\sqrt{n/\beta}\right)^{m+1}}{(1+2\kappa)(m+1)\left(1-\left(4(1+2\kappa)\theta\sqrt{n/\beta}\right)\right)} .
\end{aligned}
$$

For the remainder of this proof we assume that in the predictor step we have

$$
\theta \leq \frac{\sqrt{\beta}}{12(1+2\kappa)\sqrt{n}} \tag{4.7}
$$

which is a necessary condition for the last inequality above. In this case, since $m \geq 2$ and $\kappa \geq 0$, we deduce that

$$
\begin{aligned}
&\left\| \textstyle\sum_{i=m+1}^{2m}\theta^i h^i \right\|_2 \\
\leq\ & \tfrac{3\beta\mu}{(1+2\kappa)(m+1)}\left(4(1+2\kappa)\theta\sqrt{n/\beta}\right)^{m+1} \\
\leq\ & \beta\mu\left(4(1+2\kappa)\theta\sqrt{n/\beta}\right)^{m+1} .
\end{aligned}
$$

Since $\frac{e^Ta}{n}e$ is the projection of the vector a onto e, we have

$$
\|a - \frac{e^Ta}{n}e\| \leq \|a\| \quad and \quad \|\frac{e^Ta}{n}e\| \leq \|a\|.
$$

Therefore, $z \in \mathcal{D}(\beta)$ and (4.5) imply

$$
\begin{aligned}
& x(\theta)s(\theta) - (1-\gamma)\beta\mu(\theta) \\
=\ & (1-\theta)^{(1+\epsilon)}xs + \sum_{i=m+1}^{2m}\theta^i h^i - (1-\gamma)\beta\left[(1-\theta)^{(1+\epsilon)}\mu + \sum_{i=m+1}^{2m}\theta^i\frac{e^Th^i}{n}\right] \\
=\ & (1-\theta)^{(1+\epsilon)}(xs-\mu\beta) + \gamma\beta(1-\theta)^{(1+\epsilon)}\mu + \sum_{i=m+1}^{2m}\theta^i(h^i - \frac{e^Th^i}{n})
\end{aligned}
$$

$$
\begin{aligned}
&+(1-(1-\gamma)\beta)\sum_{i=m+1}^{2m}\theta^i\frac{e^Th^i}{n}\\
\geq\ & \gamma\beta(1-\theta)^{(1+\epsilon)}\mu-(2-(1-\gamma)\beta)\left\|\sum_{i=m+1}^{2m}\theta^i h^i\right\|_2 .
\end{aligned}
$$

Therefore the inequality

$$x(\theta)s(\theta)\geq(1-\gamma)\beta\mu(\theta)$$

holds for any θ satisfying (4.7) and

$$\gamma(1-\theta)^2\geq 2\left(4(1+2\kappa)\theta\sqrt{n/\beta}\right)^{m+1} .$$

It is easy to check that both the above inequality and (4.7) are satisfied by

$$\tilde\theta:=\frac{\sqrt{\beta}}{12(1+2\kappa)^{1+1/(m(m+1))}\sqrt{n}}\sqrt[m+1]{\gamma}\,. \tag{4.8}$$

Moreover, $\tilde\theta$ defined above also satisfies

$$\mu(\tilde\theta)\leq\left[(1-\tilde\theta)+\frac{\beta}{\sqrt{n}}\left(4(1+2\kappa)\tilde\theta\sqrt{n/\beta}\right)^{m+1}\right]\mu\ \leq(1-\frac{5}{9}\tilde\theta)\mu\,. \tag{4.9}$$

From (4.4) it follows that $\tilde\theta\leq\check\theta$, and according to (4.6) and (4.9) we have

$$\overline{\mu}\ =\mu(\overline{\theta}\,)\leq\mu(\tilde\theta)\leq\left(1-\frac{5\sqrt{\beta}\sqrt[m+1]{1-\beta}}{108(1+2\kappa)^{1+1/(m(m+1))}\sqrt{n}\sqrt[m+1]{(1+4\kappa)n+1}}\right)\mu\,. \tag{4.10}$$

Let us analyze now the corrector. From (3.35) and (4.10) we have

$$
\begin{aligned}
\mu_+\ \leq\ &\left(1+\frac{3\beta\,(1-\beta)}{2((1+4\kappa)n+1)^2}\right)\overline{\mu}\\
\leq\ &\left(1+\frac{3\beta\,(1-\beta)}{2((1+4\kappa)n+1)^2}\right)\\
&\left(1-\frac{5\sqrt{\beta}\sqrt[m+1]{1-\beta}}{108(1+2\kappa)^{1+1/(m(m+1))}\sqrt{n}\sqrt[m+1]{(1+4\kappa)n+1}}\right)\mu\\
\leq\ &\left(1-\frac{5\sqrt{\beta}\sqrt[m+1]{1-\beta}}{108(1+2\kappa)^{1+1/(m(m+1))}\sqrt{n}\sqrt[m+1]{(1+4\kappa)n+1}}\right.\\
&\left.+\frac{3\beta\,(1-\beta)}{2((1+4\kappa)n+1)^2}\right)\mu\\
\leq\ &\left(1-\frac{5\sqrt{\beta}\sqrt[3]{1-\beta}}{108(1+2\kappa)^{1+1/(m(m+1))}\sqrt{n}\sqrt[m+1]{(1+4\kappa)n+2}}\right.\\
&\left.+\frac{3\beta\,(1-\beta)}{2(1+4\kappa)n((1+4\kappa)n+2)}\right)\mu
\end{aligned}
$$

$$= \left(1 - \left(1-\overline{\lambda}\right) \frac{5\sqrt{\beta}\sqrt[3]{1-\beta}}{108(1+2\kappa)^{1+1/(m(m+1))}\sqrt{n}\sqrt[m+1]{(1+4\kappa)n+2}}\right)\mu\,,$$

where

$$\overline{\lambda} = \frac{162}{5}\frac{\sqrt{\beta}\sqrt[3]{(1-\beta)^2}}{\sqrt{n}((1+4\kappa)n+2)^{m/(m+1)}\frac{1+4\kappa}{(1+2\kappa)^{1+1/(m(m+1))}}}\,. \qquad (4.11)$$

Since

$$((1+4\kappa)n+2)^{m/(m+1)}\frac{1+4\kappa}{(1+2\kappa)^{1+1/(m(m+1))}} \geq (n+2)^{1/2},$$

and $n \geq 14$, $m \geq 2$, $\kappa \geq 0$, and $\sqrt{\beta}\sqrt[3]{(1-\beta)^2}$ is maximized for $\beta = 3/7$, we deduce that

$$\mu_+ \leq \left(1 - .0011\frac{\sqrt{\beta}\sqrt[3]{1-\beta}}{(1+2\kappa)^{1+1/(m(m+1))}\sqrt{n}\sqrt[m+1]{(1+4\kappa)n+2}}\right)\mu\,,$$

which completes the proof. ■

The next complexity result follows immediately from the above theorem,

COROLLARY 4.3. *Algorithm 2 with stopping criterion (3.9) produces a point* $z^k \in \mathcal{D}(\beta)$ *with* ${x^k}^T s^k \leq \varepsilon$ *in at most* $O\left((1+\kappa)^{1+1/m}n^{1/2+1/(m+1)}\log\left({x^0}^T s^0/\varepsilon\right)\right)$ *iterations.*

Proof The result follows immediately from the fact that

$$(1+2\kappa)^{1+1/(m(m+1))}(1+4\kappa)^{1/(m+1)} = O((1+\kappa)^{1+1/m}).$$

■

COROLLARY 4.4. *Algorithm 2 with stopping criterion (3.9) and with* $m = \Omega(\log n)$ *produces a point* $z^k \in \mathcal{D}(\beta)$ *with* ${x^k}^T s^k \leq \varepsilon$ *in at most*

$$O\left((1+\kappa)^{1+1/m}\sqrt{n}\log\left({x^0}^T s^0/\varepsilon\right)\right)$$

iterations.

Proof since $m = \Omega(\log n)$, $\exists C_1$ s.t. $m \geq C_1 \log n$, so that $n^{\frac{1}{m+1}} \leq n^{\frac{1}{C_1 \log n+1}} = C_2$, where $C_2 \leq exp(\frac{1}{C_1})$ is a constant. The result thus follows immediately from the previous corollary. ■

An obvious choice for m is $m = \lceil \log n \rceil - 1$. However, since $\lim_{n\to\infty} n^{1/n^\omega} = 1$ for any $\omega \in (0,1)$, we can choose $m = \lceil n^\omega - 1 \rceil$ for some value of $\omega \in (0,1)$. This choice was initially suggested by Roos (private communication) and subsequently used in [28] and [15]. In the following table we give the values for this choice of m with $\omega = 1/10$.

The numerical implementation of a predictor of order m requires a matrix factorization and m backsolves. If the matrices Q and R are full, the cost of a matrix factorization is $O(n^3)$ arithmetic operations, while the cost of a backsolve is $O(n^2)$ arithmetic operations. The above choices of m ensure that the cost of implementing the higher order prediction is dominated by the cost of the factorization.

Next we show that the complementarity gap of the sequence produced by Algorithm 2 with no stopping criterion is superlinearly convergent even when the

TABLE 4.1:

n	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}
$\lceil n^{.1} - 1 \rceil$	2	3	3	5	6	7	10

problem is degenerate. More precisely we have $\mu_{k+1} = O(\mu_k^{m+1})$ the HLCP (2.1) is nondegenerate, and $\mu_{k+1} = O(\mu_k^{(m+1)/2})$ otherwise. The proof is based on the following lemma which is a consequence of the results about the analyticity of the central path from [20].

LEMMA 4.5. *If HLCP (2.1) is sufficient then the solution of (4.2) satisfies*

$$u^i = O(\mu^i), \quad v^i = O(\mu^i), \quad i = 1, \ldots, m\ , \ \textit{if} \quad \textit{HLCP (2.1) is nondegenerate}\ ,$$

and

$$u^i = O(\mu^{i/2}), \quad v^i = O(\mu^{i/2}), \quad i = 1, \ldots, m\ , \ \textit{if} \ \ \textit{HLCP (2.1) is degenerate}\ .$$

By using the above lemma we can extend the superlinear convergence result of [15] to sufficient complementarity problems.

THEOREM 4.6. *The sequence μ_k produced by Algorithm 2 with no stopping criterion satisfies*

$$\mu_{k+1} = O(\mu_k^{m+1})\ , \ \textit{if} \ \ \textit{HLCP (2.1) is nondegenerate}\ , \tag{4.12}$$

and

$$\mu_{k+1} = O(\mu_k^{(m+1)/2})\ , \ \textit{if} \ \ \textit{HLCP (2.1) is degenerate}\ . \tag{4.13}$$

In order to use Algorithm 2 we fist have to find a constant κ that is greater or equal to the handicap $\chi(Q, R)$. The following algorithm does not require finding an upper bound for the handicap and therefore it can be applied to any sufficient HLCP.

Algorithm 2A
Given $\beta \in (0,\, 1)$, an integer $m \geq 2$, and $z^0 \in \mathcal{D}(\beta)$:
  Set $\mu_0 \leftarrow \mu(z_0)$, $k \leftarrow 0$, and $\kappa \leftarrow 1$;
  repeat
    Compute γ from (3.3);
    (predictor step)
    Compute directions $w^i = \lceil\, u^i,\, v^i \,\rfloor$, $\quad i = 1, \ldots, m$ by solving (4.2);
    Compute $\check{\theta}$ from (4.4);
    Compute $\overline{z}$ from (4.6) ;
    If $\mu(\overline{z}\,) = 0$ then STOP: $\overline{z}$ is an optimal solution;
    If $\overline{z} \in \mathcal{D}(\beta)$, then set $z^{k+1} \leftarrow \overline{z}$, $\mu_{k+1} \leftarrow \mu(\overline{z}\,)$, $k \leftarrow k+1$,

and RETURN;
(corrector step)
Compute centering direction (3.5) by solving (3.20);
Compute centering steplength (3.25);
Compute z^+ from (3.8);
if $z^{k+1} \in \mathcal{D}(\beta)$, set $z^{k+1} \leftarrow z^+ , \;\; \mu_{k+1} \leftarrow \mu(z^+) \, , \; k \leftarrow k+1$,
and RETURN;
else, set $\kappa \leftarrow 2\kappa$ and $z^{k+1} \leftarrow z^k , \;\; \mu_{k+1} \leftarrow \mu(z^k) \, , \; k \leftarrow k+1$,
and RETURN;
until some stopping criterion is satisfied.

By using an analysis similar to the one employed in the previous section we obtain the following result.

THEOREM 4.7. *Algorithm 2A is well defined for any sufficient HLCP and the following statements hold:*

(i) it produces a point $z^k \in \mathcal{D}(\beta)$ *with* $x^{k\,T} s^k \leq \varepsilon$ *in at most*

$$O\left(\left(1+\chi(Q,R)^{1+1/m}\right) n^{1/2+1/(m+1)} \log\left(x^{0\,T} s^0/\varepsilon\right)\right) \textit{ iterations;}$$

(ii) if we choose $m = \Omega(\log n)$*, a point* $z^k \in \mathcal{D}(\beta)$ *with* $x^{k\,T} s^k \leq \varepsilon$ *is produced in at most*

$$O\left(\left(1+\chi(Q,R)^{1+1/m}\right) \sqrt{n} \log\left(x^{0\,T} s^0/\varepsilon\right)\right) \textit{ iterations;}$$

(iii) the normalized complementarity gap satisfies (4.12) and (4.13).

5 Summary

We have presented a first order and an m'th order predictor-corrector interior-point algorithm for sufficient HLCP that depend explicitly on an upper bound κ of the handicap $\chi(Q,R)$ of the HLCP. They produce a point $\lceil x,\, s \rfloor$ in the $\mathcal{N}_\infty^-$ neighborhood of the central path with complementarity gap $x^T s \leq \varepsilon$ in at most $O\left((1+\kappa) n \log\left(x^{0\,T} s^0/\varepsilon\right)\right)$ and $O\left((1+\kappa)^{1+1/m} \sqrt{n} \log\left(x^{0\,T} s^0/\varepsilon\right)\right)$ iterations respectively. The first order method is Q-quadratically convergent for nondegenerate problems, while the m'th order method is Q-superlinearly convergent of order $m+1$ for nondegenerate problems and of order $(m+1)/2$ for degenerate problems.

We have also presented a first order and a high order predictor-corrector method for sufficient HLCP that do not require an explicit upper bound κ of the handicap $\chi(Q,R)$ and therefore can be applied to any sufficient HLCP. Their iteration complexity and superlinear convergence properties are similar to that of the previous methods with $\kappa = \chi(Q,R)$.

The cost of implementing one iteration of our algorithms is $O(n^3)$ arithmetic operations. The cost is dominated by the cost of the two matrix factorizations required both by the first order method and the high order method.

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