SIAM J. OPTIM.
Vol. 24, No. 1, pp. 1–28

INTERIOR POINT METHODS FOR SUFFICIENT HORIZONTAL LCP IN A WIDE NEIGHBORHOOD OF THE CENTRAL PATH WITH BEST KNOWN ITERATION COMPLEXITY*

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Abstract. Three interior point methods are proposed for sufficient horizontal linear complementarity problems (HLCP): a large update path following algorithm, a first order corrector-predictor method, and a second order corrector-predictor method. All algorithms produce sequences of iterates in the wide neighborhood of the central path introduced by Ai and Zhang. The algorithms do not depend on the handicap κ of the problem, so that they can be used for any sufficient HLCP. They have $O((1+\kappa)\sqrt{n}L)$ iteration complexity, the best iteration complexity obtained so far by any interior point method for solving sufficient linear complementarity problems. The first order corrector-predictor method is Q-quadratically convergent for problems that have a strict complementarity solution. The second order corrector-predictor method is superlinearly convergent with Q order 1.5 for general problems and with Q order 3 for problems that have a strict complementarity solution.

Key words. linear complementarity, interior point, path following, corrector-predictor, sufficient matrix, wide neighborhood

AMS subject classifications. 90C51, 90C33

DOI. 10.1137/120884341

1. Introduction.

The notion of a sufficient linear complementarity problem was introduced in 1989 by Cottle, Pang, and Venkateswaran [5]. Two years later Kojima, Megiddo, Noma, and Yoshise [12] introduced the notion of a P_* linear complementarity problem and showed that a P_*-matrix is column sufficient. Subsequently, Guu and Cottle [8] proved that a P_*-matrix is also row sufficient and therefore the class P_* is included in the class of sufficient matrices. Soon after that Väliaho [27] proved the reverse inclusion. Therefore P_* coincides with the class of sufficient matrices. Since $P_* = \cup_{\kappa \geq 0} P_*(\kappa)$, it follows that a matrix is sufficient if and only if it is a $P_*(\kappa)$-matrix for some $\kappa \geq 0$. The smallest κ with this property is called the handicap of the matrix. Kojima, Megiddo, Noma, and Yoshise introduced the class P_* because many interior point methods, originally developed for linear programming, can be extended in a natural way to P_* linear complementarity problem. For example, they proved that the primal-dual potential reduction method can solve a $P_*(\kappa)$ linear complementarity problem in at most $O((1+\kappa)\sqrt{n}L)$ iterations, where n is the dimension of the problem and $L = \log(\varepsilon_0/\varepsilon)$ with ε the required precision (duality gap) and ε_0 the duality gap at the starting point. This is still the best complexity result for solving $P_*(\kappa)$ linear complementarity problems. No superlinear complexity results were given for the interior point methods from [12]. In 1995 Miao [15] extended the MTY [16] predictor-corrector method for $P_*(\kappa)$ linear complementarity problems. His algorithm uses the l_2 neighborhood of the central path, has $O((1+\kappa)\sqrt{n}L)$ iteration complexity, and is quadratically convergent for nondegenerate problems (i.e., problems that admit a strict complementarity solution). However, the constant κ is

*Received by the editors July 12, 2012; accepted for publication (in revised form) September 5, 2013; published electronically January 2, 2014.
http://www.siam.org/journals/siopt/24-1/88434.html

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explicitly used in the construction of the algorithm. It is well known that this constant is very difficult to estimate for many sufficient linear complementarity problems. The predictor-corrector methods described in [20] do not depend on the constant κ, so that the same algorithm can be used for any sufficient complementarity problem. The sufficiency property cannot be verified in polynomial time. Interior point methods for sufficient complementarity problems have been modified according to the duality theorem in existentially polynomial-time form in such a way that when applied to a general linear complementarity problem, in polynomial time they either give a solution of the original problem or detect the lack of the $P_*(\kappa)$ property with arbitrarily large but a priori fixed κ [9, 10, 11].

By employing a higher order predictor, arbitrarily high orders of convergence on nondegenerate problems can be attained. Finally, by using the "fast-safe-improve" strategy of Wright and Zhang [28], the algorithms of [20] require asymptotically only one matrix factorization per iteration, while Miao's algorithm requires two matrix factorizations at every iteration. The predictor-corrector algorithms presented in [21, 22] are superlinearly convergent even for degenerate problems. More precisely the Q-order of convergence of the complementarity gap is 2 for nondegenerate problems and 1.25 for degenerate problems. The interior point methods from [20, 21, 22] use a neighborhood that is slightly larger than the l_2 neighborhood (but smaller than the l_∞^- neighborhood) and have $O((1+\kappa)\sqrt{n}L)$ iteration complexity.

Predictor-corrector algorithms with arbitrarily high order of convergence for degenerate sufficient linear complementarity problems were given in [26]. The algorithms depend on the constant κ, use an l_2 neighborhood of the central path, and, as shown in [25], have $O((1+\kappa)\sqrt{n}L)$ iteration complexity for $P_*(\kappa)$ linear complementarity problems. A general local analysis of higher order predictor-corrector methods in an l_2 neighborhood for degenerate sufficient linear complementarity problems is given by Zhao and Sun [29], who also propose a new algorithm that does not need a corrector step. The latter algorithm does not follow the traditional central path. Instead a new analytic path is used at each iteration. No complexity results are given.

Although some of the above mentioned algorithms have optimal iteration complexity, they are not as efficient in practice as the algorithms that use a wide neighborhood of the central path and have worse iteration complexity. This well-known discrepancy between theory and practice has been mitigated by a series of results obtained in the last decade. By using a high order corrector-predictor approach Liu and Potra [14] have managed to obtain an interior point method acting in the wide l_∞^- neighborhood of the central path that does not depend on the handicap κ of the problem, has $O((1+\kappa)\sqrt{n}L)$ iteration complexity, and is superlinearly convergent for general sufficient linear complementarity problems. The algorithm uses two matrix factorizations and $\log n$ backsolves per iteration. A class of interior point methods with similar properties that use only one matrix factorization per iteration is analyzed in [23].

Another way of improving the complexity of interior point methods acting in wide neighborhoods of the central path is to consider wide neighborhoods that are different from the classical l_∞^- neighborhood. For example, in the monograph [17] Peng, Roos, and Terlaky consider wide neighborhoods defined by means of self-regular functions. They propose an interior point algorithm for solving a class of $P_*(\kappa)$ nonlinear linear complementarity problems based on such wide neighborhoods. Their algorithm does not depend on κ. They establish the complexity of the algorithm in terms of n only, by tacitly assuming that κ is a finite constant. By analyzing the proof of Theorem 4.4.10 of the monograph it follows that the iteration complexity of their algorithm (with $q = \log n$) when applied to a $P_*(\kappa)$ linear complementarity problem is $O((1 + \kappa) \log n \sqrt{n}L)$. Interior point methods for sufficient linear complementarity problems based on neighborhoods defined by so-called eligible kernel functions, which are not necessarily self-regular, are considered in [3, 13]. The algorithms from [3, 13] depend explicitly on κ and have $O((1 + \kappa) \log n \sqrt{n}L)$-iteration complexity for appropriate choices of kernel functions.

In the present paper we describe and analyze three interior point methods for sufficient horizontal linear complementarity problems (HLCP) acting in the wide neighborhood $\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ of the central path proposed by Ai and Zhang [1]. This neighborhood is wide in the sense that for appropriate choice of the parameter γ it contains any given l^-_∞ neighborhood of the central path. Our results extend in a nontrivial way the interior point methods developed in [1] for monotone, i.e., $P_*(0)$, linear complementarity problems. The algorithms do not depend on the handicap κ of the problem, so that they can be used for any sufficient HLCP. They have $O((1 + \kappa)\sqrt{n}L)$ iteration complexity, the best iteration complexity obtained so far by any interior point method for solving sufficient linear complementarity problems. This improves the best previously known $O((1 + \kappa) \log n \sqrt{n}L)$ iteration complexity result for interior point methods acting in a wide neighborhood of the central path that use only a fixed number (independent of n) of backsolves per iteration.

We first present a large update path following algorithm that generalizes the corresponding algorithm from [1]. The algorithm requires the solution of a two-dimensional optimization problem. We also give a variant of this algorithm with the same iteration complexity which uses a simple numerical scheme for finding an approximate solution of the two-dimensional optimization problem. We note that Ai and Zhang also propose a practical implementation of the corresponding two-dimensional optimization problem [1, Algorithm 4.3.]. Their approach cannot be generalized directly to sufficient HLCP without explicit use of the handicap κ. Ai and Zhang [1] also describe a predictor-corrector method for solving monotone LCP that is Q-quadratically convergent for problems admitting a strict complementarity solution. Since the predictor-corrector approach cannot be generalized to sufficient HLCP without explicit use of the handicap, we employ the corrector-predictor approach originally introduced in [18]. Our first order corrector-predictor method is Q-quadratically convergent for problems that have a strict complementarity solution. Finally, we propose a second order corrector-predictor method that is superlinearly convergent with Q order 1.5 for problems that do not have a strict complementarity solution and with Q order 3 for problems that have a strict complementarity solution. We note that the interior point methods from [1] use the neighborhood $\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ only for $\alpha \in (0, 1/2]$, while our algorithms use this neighborhood for any value of $\alpha \in (0, 1)$.

Conventions. We denote by $\mathbb{N}$ the set of all nonnegative integers. $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$ denote the set of real, nonnegative real, and positive real numbers, respectively. For any real number κ, $\lceil \kappa \rceil$ denotes the smallest integer greater than or equal to κ. Given a vector x, the corresponding uppercase symbol denotes, as usual, the diagonal matrix X defined by the vector. The symbol e represents the vector of all ones, with dimension given by the context.

We denote componentwise operations on vectors by the usual notation for real numbers. Thus, given two vectors u, v of the same dimension, uv, u/v, etc., will denote the vectors with components $u_i v_i$, u_i/v_i, etc. This notation is consistent as long as componentwise operations always have precedence in relation to matrix operations. Note that $uv \equiv Uv$ and if A is a matrix, then $Auv \equiv AUv$, but in general $Auv \neq (Au)v$. Also if f is a scalar function and v is a vector, then $f(v)$ denotes the vector with components $f(v_i)$. For example, if $v \in \mathbb{R}^n_+$, then $\sqrt{v}$ denotes the vector with components $\sqrt{v_i}$, and $1 - v$ denotes the vector with components $1 - v_i$. Traditionally the vector $1 - v$ is written as $e - v$, where e is the vector of all ones. Inequalities are to be understood in a similar fashion. For example, if $v \in \mathbb{R}^n$, then $v \geq 3$ means that $v_i \geq 3,\ i = 1, \ldots, n$. Traditionally this is written as $v \geq 3\,e$. If $\|\,.\,\|$ is a vector norm on $\mathbb{R}^n$ and A is a matrix, then the operator norm induced by $\|\,.\,\|$ is defined by $\|\,A\,\| = \max\{\|\,Ax\,\|\ ;\ \|\,x\,\| = 1\}$. As a particular case we note that if U is the diagonal matrix defined by the vector u, then $\|\,U\,\|_2 = \|\,u\,\|_\infty$. The positive and negative parts of a vector $v \in \mathbb{R}^n$ are defined by $v^+ = \max\{v, 0\}$, $v^- = \min\{v, 0\}$, so that $v^+ \geq 0$, $v^- \leq 0$, and $v = v^+ + v^-$.

We use the notation $O(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$, and $o(\cdot)$ in the standard way to express asymptotic relationships between functions. The most common usage will be associated with a sequence $\{x^k\}$ of vectors and a sequence $\{\tau_k\}$ of positive real numbers. In this case $x^k = O(\tau_k)$ means that there is a constant K (dependent on problem data) such that for every $k \in \mathbb{N}$, $\left\|\,x^k\,\right\| \leq K\tau_k$. Similarly, if $x^k > 0$, $x^k = \Omega(\tau_k)$ means that $(x^k)^{-1} = O(1/\tau_k)$. If we have both $x^k = O(\tau_k)$ and $x^k = \Omega(\tau_k)$, we write $x^k = \Theta(\tau_k)$

If $x, s \in \mathbb{R}^n$, then the vector $z \in \mathbb{R}^{2n}$ obtained by concatenating x and s will be denoted by $[\,x\,;\,s\,]$, i.e.,

$$z = [\,x\,;\,s\,] = \begin{bmatrix} x \\ s \end{bmatrix} = \left[\,x^T,\ s^T\,\right]^T. \tag{1.1}$$

Throughout this paper the mean value of xs will be denoted by

$$\mu = \mu(z) = \frac{x^T s}{n}. \tag{1.2}$$

2. The $P_*(\kappa)$ HLCP. Given two matrices $Q, R \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$, the HLCP consists in finding a pair of vectors $z = [\,x\,;\,s\,]$ such that

$$\begin{aligned} xs &= 0, \\ Qx + Rs &= b, \\ x, s &\geq 0. \end{aligned} \tag{2.1}$$

The standard (monotone) linear complementarity problem is obtained by taking $R = -I$ and Q positive semidefinite. Let $\kappa \geq 0$ be a given constant. We say that (2.1) is a $P_*(\kappa)$ HLCP if

$$Qu + Rv = 0 \text{ implies } (1 + 4\kappa)\sum_{i\in\mathcal{I}^+} u_i v_i + \sum_{i\in\mathcal{I}^-} u_i v_i \geq 0 \text{ for any } u, v \in \mathbb{R}^n,$$

where $\mathcal{I}^+ = \{i : u_i v_i > 0\}$ and $\mathcal{I}^- = \{i : u_i v_i < 0\}$. If the above condition is satisfied we say that (Q, R) is a $P_*(\kappa)$ pair and we write $(Q, R) \in P_*(\kappa)$. In case $R = -I$, $(Q, -I)$ is a $P_*(\kappa)$ pair if and only if Q is a $P_*(\kappa)$-matrix in the sense that

$$(1 + 4\kappa)\sum_{i\in\hat{\mathcal{I}}^+} x_i[Qx]_i + \sum_{i\in\hat{\mathcal{I}}^-} x_i[Qx]_i \geq 0 \quad \forall x \in \mathbb{R}^n,$$

where $\hat{\mathcal{I}}^+ = \{i : x_i[Qx]_i > 0\}$ and $\hat{\mathcal{I}}^- = \{i : x_i[Qx]_i < 0\}$. Problem (2.1) is then called a $P_*(\kappa)$ linear complementarity problem and it is extensively discussed in [12]. If (Q, R) belongs to the class

$$P_* = \cup_{\kappa\geq 0} P_*(\kappa)\,,$$

then we say that (Q, R) is a P_* pair and (2.1) is a P_* HLCP.

The class of sufficient matrices was defined by Cottle, Pang, and Venkateswaran in [5]. The appropriate generalization to sufficient pairs [25, 26] is in terms of the null space of the matrix $[Q\ R] \in \mathbb{R}^{n\times 2n}$

$$\Phi := \mathcal{N}([Q\ R]) = \{[\,u\,;\,v\,]\,|Qu + Rv = 0\} \tag{2.2}$$

and its orthogonal space

$$\Phi^{\perp} = \big\{[\,u\,;\,v\,]\ :\ u = Q^T x,\, v = R^T x,\ \text{ for some } x \in \mathbb{R}^n\big\}. \tag{2.3}$$

A pair (Q, R) is called column sufficient if

$$[\,u\,;\,v\,] \in \Phi, \quad uv \le 0 \text{ implies } uv = 0,$$

and row sufficient if

$$[\,u\,;\,v\,] \in \Phi^{\perp}, \quad uv \ge 0 \text{ implies } uv = 0.$$

(Q, R) is a sufficient pair if it is both column and row sufficient. The corresponding results of row and column sufficient matrices in [4] can be extended to row and column sufficient pairs (see, for example, [24]): (Q, R) is a sufficient pair if and only if HLCP (2.1) has a convex (perhaps empty) solution set for any b, and every KKT point of

$$\begin{array}{cc} \text{minimize}_{x,s} & x^T s \\ \text{subject to} & Qx + Rs = b, \\ & x, s \ge 0 \end{array}$$

is a solution of (2.1).

Väliaho's result [27] states that a matrix is sufficient if and only if it is a $P_*(\kappa)$ matrix for some $\kappa \ge 0$. The result can be extended to sufficient pairs by using the equivalence results from [2] (see also [24]): (Q, R) is a sufficient pair if and only if there is a finite $\kappa \ge 0$ so that (Q, R) is a $P_*(\kappa)$ pair. By extension, a P_* HLCP will be called a sufficient HLCP and a P_* pair will be called a sufficient pair.

Let us note that if (Q, R) is a sufficient pair, then the matrix $[Q\ R]$ has full rank. This is a consequence of the following slightly stronger result proved in [14].

THEOREM 2.1. *If the pair (Q, R) is column sufficient, then the matrix $[Q\ R]$ has full rank.*

A simple example given in [14] shows that row sufficiency alone does not imply the full rank property. We also note that in [24, 26, 25] the full rank property was given as an assumption, which in fact always holds because of the above theorem.

We denote the set of all feasible points of HLCP by

$$\mathcal{F} = \{z = [\,x\,;\,s\,] \in \mathbb{R}^{2n}_{+}\ :\quad Qx + Rs = b\} \tag{2.4}$$

and its solution set by

$$\mathcal{F}^* = \{z^* = [\,x^*\,;\,s^*\,] \in \mathcal{F}\ :\quad x^* s^* = 0\}. \tag{2.5}$$

The relative interior of $\mathcal{F}$, which is also known as the set of strictly feasible points or the set of interior points, is given by

$$\mathcal{F}^0 = \mathcal{F} \bigcap \mathbb{R}^{2n}_{++}. \tag{2.6}$$

It is known (see, for example, [12]) that if $\mathcal{F}^0$ is nonempty, then the nonlinear system

$$\begin{aligned} xs &= \tau e, \\ Qx + Rs &= b \end{aligned}$$

with sufficient (Q, R) pair has a unique positive solution for any $\tau > 0$. The set of all such solutions defines the central path $\mathcal{C}$ of the HLCP, that is,

$$\mathcal{C} = \{z \in \mathbb{R}^{2n}_{++} \; : \; F_\tau(z) = 0, \, \tau > 0\},$$

where

$$F_\tau(z) = \left[\begin{array}{c} xs - \tau e \\ Qx + Rs - b \end{array} \right].$$

If $F_\tau(z) = 0$, then it is easily seen that $\tau = \mu = \mu(z)$, where $\mu(z)$ is given by (1.2). The wide neighborhood of the central path $\mathcal{N}^-_\infty(\alpha)$ is defined as

$$\mathcal{N}^-_\infty(\alpha) = \{z \in \mathcal{F}^0 \; : \; \delta^-_\infty(z) \le \alpha \},$$

where $0 < \alpha < 1$ is a given parameter and

$$\delta^-_\infty(z) := \left\| \left[\frac{xs}{\mu} - e \right]^- \right\|_\infty$$

is a proximity measure of z to the central path. Alternatively, if we denote

$$\mathcal{D}(\beta) = \{z \in \mathcal{F}^0 : xs \ge \beta\mu(z)\},$$

then the neighborhood $\mathcal{N}^-_\infty(\alpha)$ can also be written as

$$\mathcal{N}^-_\infty(\alpha) = \mathcal{D}(1 - \alpha).$$

It is well known (see, for example, [18]) that

$$\lim_{\alpha\downarrow 0} \mathcal{N}^-_\infty(\alpha) = \lim_{\beta\uparrow 1} \mathcal{D}(\beta) = \mathcal{C}, \;\; \lim_{\alpha\uparrow 1} \mathcal{N}^-_\infty(\alpha) = \lim_{\beta\downarrow 0} \mathcal{D}(\beta) = \mathcal{F}.$$

In this paper we will work with the following neighborhood considered in [1]:

$$\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha) = \left\{ z \in \mathcal{F}^0 \; : \; \widetilde{\delta}^-_{2,\gamma}(z) \le \alpha \right\}, \quad \widetilde{\delta}^-_{2,\gamma}(z) := \left\| \left[\frac{xs}{\gamma\mu} - e \right]^- \right\|_2 . \tag{2.7}$$

It is easily verified that $\widetilde{\delta}^-_{2,\gamma}(z) = 0 \; \forall z \in \mathcal{D}(\gamma)$ and that for any $z = [\, x \, ; \, s \,]$

$$\mu > 0 \,\&\, \widetilde{\delta}^-_{2,\gamma}(z) \le \alpha \,\&\, x_i s_i \le \gamma\mu \Rightarrow 0 \le 1 - \frac{x_i s_i}{\gamma\mu} \le \alpha \Rightarrow x_i s_i \ge (1-\alpha)\gamma\mu. \tag{2.8}$$

Therefore we have

$$\mathcal{D}(\gamma) \subset \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha) \subset \mathcal{D}\left((1-\alpha)\gamma\right) \quad \forall \alpha, \gamma \in (0,1). \tag{2.9}$$

Since $\mathcal{D}(\gamma)$ is a wide neighborhood, so is $\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$. We note that the interior point methods from [1] use the neighborhood $\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ only for $\alpha \in (0, 1/2]$, while in this paper we will construct interior point methods based on the neighborhood $\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ for any value of $\alpha \in (0, 1)$.

3. A large update path following algorithm. In this section we present a large update path following method for solving the sufficient HLCP (2.1). Our algorithm generalizes the large update path following method proposed in [1] for standard monotone linear complementarity problems. Unlike the algorithms from [3, 13], our method does not depend on the handicap of the problem and therefore can be applied to any sufficient HLCP.

At a typical iteration of the algorithm we are given a point $z \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ and we compute the vectors $w^1 = [u^1; v^1], w^2 = [u^2; v^2]$ by solving the linear systems

$$\begin{cases} su^1 + xv^1 = (\gamma\mu e - xs)^-, \\ Qu^1 + Rv^1 = 0, \end{cases} \tag{3.1}$$

$$\begin{cases} su^2 + xv^2 = (\gamma\mu e - xs)^+, \\ Qu^2 + Rv^2 = 0. \end{cases} \tag{3.2}$$

For any $\theta = (\theta_1, \theta_2)$ we consider the vector

$$w(\theta) = [\, u(\theta)\, ;\, v(\theta)\,] = \theta_1 w^1 + \theta_2 w^2, \tag{3.3}$$

whose components satisfy the linear system

$$\begin{cases} su(\theta) + xv(\theta) = \theta_1(\gamma\mu e - xs)^- + \theta_2(\gamma\mu e - xs)^+, \\ Qu(\theta) + Rv(\theta) = 0 \end{cases} \tag{3.4}$$

and we denote

$$z(\theta) = [\, x(\theta)\, ;\, s(\theta)\,] = z + w(\theta), \quad \mu(\theta) = x(\theta)^T s(\theta)/n. \tag{3.5}$$

The two-dimensional vector θ^+ of steplengths along the directions w^1, w^2 is obtained by solving the two-dimensional optimization problem

$$\begin{array}{ll} \text{minimize}_{\,\theta\in[0,1]\times[01]} & \mu(\theta), \\ \text{subject to} & z(\theta) \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha). \end{array} \tag{3.6}$$

It turns out that the constraint $z(\theta) \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ from the above optimization problem is equivalent to $\mu(\theta) > 0$ and $\widetilde{\delta}^{-}_{2,\gamma}(z(\theta)) \le \alpha$. This fact follows from the results of [1], but for the sake of completeness we will give a short proof.

PROPOSITION 3.1. *If*

$$[\, x\, ;\, s\,] \in R^{2n}_{++},\ [\, u\, ;\, v\,] \in \mathbb{R}^{2n},\ \rho \in (0,1],\ su + xv + xs \ge 0,\ (x+\rho u)(s+\rho v) > 0,$$

then $x + tu > 0,\ s + tv > 0\ \forall t \in (0, \rho]$.

Proof. By denoting $r = su + xv + xs$, we have

$$\begin{aligned} \rho^2(x+tu)(s+tv) &= \rho^2\left(xs + t(su+xv) + t^2 uv\right) = \rho^2((1-t)xs + tr + t^2 uv) \\ &= \rho^2(1-t)xs + \rho^2 tr + t^2\left((x+\rho u)(s+\rho v) - (1-\rho)xs - \rho r\right) \\ &= (\rho^2(1-t) - t^2(1-\rho))xs + (\rho^2 t - t^2\rho)r + t^2(x+\rho u)(s+\rho v) \\ &\ge t^2(x+\rho u)(s+\rho v) > 0\ \forall t \in (0,\rho]. \end{aligned}$$

If there is i and $\bar{t} \in (0,\rho]$ such that either $x_i + \bar{t}\, u_i \le 0$ or $s_i + \bar{t}\, v_i \le 0$, then there is $t \in (0, \bar{t}\,]$ such that either $x_i + tu_i = 0$ or $s_i + tv_i = 0$ which contradicts the fact that $(x+tu)(s+tv) > 0$. Hence the conclusion of our proposition must hold. ∎

Corollary 3.2. *Let $z \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ and let $z(\theta)$ be defined by (3.1)–(3.5). Then $z(\theta) \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ if and only if $\mu(\theta) > 0$ and $\widetilde{\delta}^{-}_{2,\gamma}(z(\theta)) \le \alpha$.*

Proof. Let us assume that $\mu(\theta) > 0$ and $\widetilde{\delta}^{-}_{2,\gamma}(z(\theta)) \le \alpha$. Since $Qu(\theta)+Rv(\theta) = 0$, we have $Qx(\theta)+Rs(\theta) = Qx+Rs = b$. In order to complete the proof of our corollary we have to show that $x(\theta) > 0$ and $s(\theta) > 0$. From (2.8) we deduce that $x(\theta)s(\theta) > 0$. According to Proposition 3.1, it is sufficient to prove that $su(\theta) + xv(\theta) + xs \ge 0$. From (3.4) it follows that the left-hand side of this inequality is equal to

$$xs + \theta_1(\gamma\mu e - xs)^- + \theta_2(\gamma\mu e - xs)^+. \tag{3.7}$$

If $x_i s_i \le \gamma\mu$, then the ith component of (3.7) becomes $x_i s_i + \theta_2(\gamma\mu - x_i s_i)$, which is clearly positive. On the other hand, if $x_i s_i > \gamma\mu$, then the ith component of (3.7) reduces to $x_i s_i + \theta_1(\gamma\mu - x_i s_i)$, which is also positive. ∎

It follows that the optimization problem (3.6) is equivalent to

$$\begin{array}{ll} \text{minimize}_{\,\theta\in[0,1]\times[01]} & \mu(\theta) \\ \text{subject to} & \mu(\theta) > 0 \text{ and } \widetilde{\delta}^{-}_{2,\gamma}(z(\theta)) \le \alpha. \end{array} \tag{3.8}$$

If $\theta^+ = (\theta_1^+, \theta_2^+)$ is the solution of the above minimization problem, then according to Corollary 3.2 the point

$$z^+ = z(\theta^+) \tag{3.9}$$

belongs to the wide neighborhood $\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ and the process can be iterated. We have thus obtained the following iterative procedure.

Algorithm 1.

Given parameters $\alpha \in (0,1)$, $\gamma \in (0, 0.5]$, and starting point $z^0 \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$:
  Set $k \leftarrow 0$;
  repeat
    Set $z = [\,x\,;\,s\,] \leftarrow z^k = [\,x^k\,;\,s^k\,]$, $\mu \leftarrow \mu_k$;
    Compute directions $w^i = [\,u^i\,;\,v^i\,]$, $i = 1, 2$, from (3.1) and (3.2);
    Consider the notation from (3.3) and (3.5);
    Compute steplength θ^+ as the solution of (3.8);
    Compute z^+ from (3.9);
    Set $\theta_k \leftarrow \theta^+$, $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow x^{+T}s^+/n$;
    Set $k \leftarrow k+1$.
  continue

In the next subsection we will analyze the properties of the large update path following algorithm described above. For simplicity we will assume that θ^+ is the exact solution of the minimization problem (3.8). However, in practice it is possible to implement the algorithm by using an approximate solution of this two-dimensional nonlinear programming problem with preservation of all properties of the algorithm, as shown in subsection 3.2.

3.1. Polynomiality of the large update algorithm. In the proof of the polynomial complexity of Algorithm 1 we will often use the following simple results.

LEMMA 3.3. *For any vectors $u, v \in \mathbb{R}^n$ and we have*

$$\begin{aligned}
&\|\,(u+v)^+\,\|_p \le \|\,u^+\,\|_p + \|\,v^+\,\|_p\,,\ \|\,(u+v)^-\,\|_p \le \|\,u^-\,\|_p + \|\,v^-\,\|_p\,,\\
&u \le v \Rightarrow \|\,u^+\,\|_p \le \|\,v^+\,\|_p \ \&\ \|\,u^-\,\|_p \ge \|\,v^-\,\|_p\,,\\
&u \ge 0 \ \&\ v \ge 0 \Rightarrow \|\,(u-v)^+\,\|_p \le \|\,u\,\|_p \ \&\ \|\,(u-v)^-\,\|_p \le \|\,v\,\|_p\,,\\
&u \ge 0 \ \&\ v \ge 0 \Rightarrow \|\,(u-v)^-\,\|_2^2 \le \left(\|\,v\,\|_2^2 - (\min u)^2\right)^+.
\end{aligned}$$

Proof. We only give the proof of the last statement. Assume that $u \ge 0$ and $v \ge 0$. We denote, just for this proof, $\mathcal{J}^- = \{i \,:\, u_i < v_i\}$. If $\mathcal{J}^-$ is empty there is nothing to prove. Therefore we may assume that $\mathrm{card}(\mathcal{J}^-) \ge 1$ and we can write

$$\begin{aligned}
\|\,(u-v)^-\,\|_2^2 &= \sum_{i\in\mathcal{J}^-}(u_i^2 - 2u_iv_i + v_i^2) \le \sum_{i\in\mathcal{J}^-}(v_i^2 - u_i^2)\\
&\le \|\,v\,\|_2^2 - \sum_{i\in\mathcal{J}^-} u_i^2 \le \|\,v\,\|_2^2 - \mathrm{card}(\mathcal{J}^-)(\min u)^2\\
&\le \left(\|\,v\,\|_2^2 - (\min u)^2\right)^+. \qquad \square
\end{aligned}$$

We will also use different bounds for the solution of a linear system of the form

$$\begin{cases} su + xv &= a,\\ Qu + Rv &= 0. \end{cases} \tag{3.10}$$

Let us introduce the notation

$$D = X^{-1/2}S^{1/2}, \quad \|\,[\,u\,;\,v\,]\,\|_z^2 := \|Du\|_2^2 + \|D^{-1}v\|_2^2, \quad \tilde{a} = (xs)^{-1/2}a. \tag{3.11}$$

Using Lemma 3.1 of [7] and its proof we obtain the following bounds.

LEMMA 3.4. *If HLCP is $P_*(\kappa)$, then for any $z = [\,x\,;\,s\,] \in \mathbb{R}^{2n}_{++}$ and any $a \in \mathbb{R}^n$ the linear system* (3.10) *has a unique solution $w = [\,u\,;\,v\,]$ for which the following estimates hold:*

$$\|\,w\,\|_z^2 \le (1+2\kappa)\|\tilde{a}\|_2^2,\ \|\,uv\,\|_2 \le \left(\frac{1}{\sqrt{8}} + \kappa\right)\|\,\tilde{a}\,\|_2^2\,,\ \|\,uv\,\|_\infty \le \left(\frac{1}{4} + \kappa\right)\|\,\tilde{a}\,\|_2^2\,,$$

$$\|\,[uv]^+\,\|_1 \le \frac{1}{4}\|\,\tilde{a}\,\|_2^2\,,\ \|\,[uv]^-\,\|_1 \le \left(\frac{1}{4} + \kappa\right)\|\,\tilde{a}\,\|_2^2\,,\ -\kappa\|\,\tilde{a}\,\|_2^2 \le u^Tv \le \frac{1}{4}\|\,\tilde{a}\,\|_2^2\,.$$

COROLLARY 3.5. *The solutions of* (3.1), (3.2), *and* (3.4) *satisfy the following inequalities:*

$$-\kappa\mu \le \frac{u^{1T}v^1}{n} \le \frac{\mu}{4}, \quad -\kappa\frac{\gamma\alpha^2\mu}{n(1-\alpha)} \le \frac{u^{2T}v^2}{n} \le \frac{\gamma\alpha^2\mu}{4n(1-\alpha)}\,, \tag{3.12}$$

$$\|\,u^1v^1\,\|_2 \le \left(\frac{1}{\sqrt{8}} + \kappa\right)n\mu,\ \|\,u^2v^2\,\|_2 \le \left(\frac{1}{\sqrt{8}} + \kappa\right)\frac{\gamma\alpha^2\mu}{1-\alpha}, \tag{3.13}$$

$$-\kappa\left(\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{n(1-\alpha)}\right)\mu \le \frac{u(\theta)^Tv(\theta)}{n} \le \left(\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{n(1-\alpha)}\right)\frac{\mu}{4}\,, \tag{3.14}$$

$$\|\,u(\theta)v(\theta)\,\|_2 \le \left(\frac{1}{\sqrt{8}} + \kappa\right)\left(n\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{1-\alpha}\right)\mu\,\cdot \tag{3.15}$$

Proof. For any $z = [\,x\,;\,s\,] \in \mathbb{R}^{2n}_{++}$ there holds

$$\left\| (xs)^{-1/2}(\gamma\mu e - xs)^- \right\|_2^2 = \left\| (xs)^{-1/2}(xs - \gamma\mu e)^+ \right\|_2^2 \leq \left\| (xs)^{1/2} \right\|_2^2 = n\mu.$$

If $z \in \widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$, then according to (2.9) we can write

$$\left\| (xs)^{-1/2}(\gamma\mu e - xs)^+ \right\|_2^2 \leq \frac{\|\,(\gamma\mu e - xs)^+\,\|_2^2}{(1-\alpha)\gamma\mu} \leq \frac{\gamma\mu\alpha^2}{1-\alpha}\,.$$

Therefore, using the orthogonality of $(\gamma\mu e - xs)^-$ and $(\gamma\mu e - xs)^+$ we have

$$\begin{aligned}&\left\| (xs)^{1/2}\left(\theta_1(\gamma\mu e - xs)^- + \theta_2(\gamma\mu e - xs)^+\right) \right\|_2^2 \\ &\quad = \theta_1^2 \left\| (xs)^{1/2}(\gamma\mu e - xs)^- \right\|_2^2 + \theta_2^2 \left\| (xs)^{1/2}(\gamma\mu e - xs)^+ \right\|_2^2 \leq n\mu\theta_1^2 + \frac{\gamma\mu\alpha^2\theta_2^2}{1-\alpha}\,.\end{aligned}$$

The desired inequalities are then obtained by using Lemma 3.4. ◻

In the next lemmas we give upper and lower bounds for the normalized duality gap $\mu(\theta)$ and for the norm of the vector $x(\theta)s(\theta)$ (see (3.5)). We first note that

$$x(\theta)s(\theta) = xs + \theta_1(\gamma\mu e - xs)^- + \theta_2(\gamma\mu e - xs)^+ + u(\theta)v(\theta), \tag{3.16}$$

$$\mu(\theta) = \mu + \theta_1 e^T(\gamma\mu e - xs)^-/n + \theta_2 e^T(\gamma\mu e - xs)^+/n + u(\theta)^T v(\theta)/n. \tag{3.17}$$

LEMMA 3.6. *For any $z \in \widetilde{\mathcal{N}}^-_{\gamma}(\alpha)$ and $\theta = [\,\theta_1\,;\,\theta_2\,]$ with $0 \leq \theta_1 \leq \theta_2 \leq 1$, we have*

$$\mu(\theta) \leq \left(1 - \left(1 - \gamma + \frac{\alpha\gamma}{\sqrt{n}}\right)\theta_1 + \frac{\gamma\alpha\theta_2}{\sqrt{n}} + \frac{\theta_1^2}{4} + \frac{\gamma\alpha^2\theta_2^2}{4n(1-\alpha)}\right)\mu, \tag{3.18}$$

$$\mu(\theta) \geq \left(1 - (1-\gamma)\,\theta_1 - \kappa\left(\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{n(1-\alpha)}\right)\right)\mu. \tag{3.19}$$

Proof. Since

$$n(1-\gamma)\mu = e^T(xs - \gamma\mu e) = e^T(xs - \gamma\mu e)^+ + e^T(xs - \gamma\mu e)^-$$

and

$$|e^T(xs - \gamma\mu e)^-| \leq \sqrt{n}\,\|\,(xs - \gamma\mu e)^-\,\|_2 \leq \gamma\alpha\mu\sqrt{n}, \tag{3.20}$$

it follows that

$$n(1-\gamma)\mu \leq e^T(xs - \gamma\mu e)^+ \leq \left(n(1-\gamma) + \gamma\alpha\sqrt{n}\right)\mu. \tag{3.21}$$

From (3.17) and the above inequalities we obtain

$$\begin{aligned}\mu(\theta) &= \mu - \theta_1\frac{e^T(xs - \gamma\mu e)^+}{n} - \theta_2\frac{e^T(xs - \gamma\mu e)^-}{n} + \frac{u(\theta)^T v(\theta)}{n} \\ &= \mu - \theta_1(1-\gamma)\mu - (\theta_2 - \theta_1)\frac{e^T(xs - \gamma\mu e)^-}{n} + \frac{u(\theta)^T v(\theta)}{n} \\ &\leq \mu - \theta_1(1-\gamma)\mu + (\theta_2 - \theta_1)\frac{\alpha\gamma\mu}{\sqrt{n}} + \frac{u(\theta)^T v(\theta)}{n} \\ &= \mu - \theta_1\left(1 - \gamma + \frac{\alpha\gamma}{\sqrt{n}}\right)\mu + \theta_2\frac{\alpha\gamma\mu}{\sqrt{n}} + \frac{u(\theta)^T v(\theta)}{n}\,.\end{aligned} \tag{3.22}$$

Similarly, we have

$$\mu(\theta) = \mu - \theta_1(1-\gamma)\mu - (\theta_2-\theta_1)\frac{e^T(xs-\gamma\mu e)^-}{n} + \frac{u(\theta)^T v(\theta)}{n}$$
$$\geq \mu - \theta_1(1-\gamma)\mu + \frac{u(\theta)^T v(\theta)}{n}\cdot$$

The required inequalities then follow from (3.14). ∎

LEMMA 3.7. *For any $z \in \widetilde{\mathcal{N}}_\gamma^-(\alpha)$ and $\theta = (\theta_1, \theta_2) \in [0,1]\times[0,1]$ satisfying*

$$\frac{\alpha\gamma\theta_2}{(1-\gamma)\sqrt{n}+\alpha\gamma} \leq \theta_1 \leq \theta_2 \leq 1, \tag{3.23}$$

there holds

$$\left\| (x(\theta)s(\theta) - \gamma\mu(\theta)e)^- \right\|_2 \leq \left((1-\theta_2)\,\gamma\alpha + \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(n\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{1-\alpha}\right)\right)\mu.$$

Proof. From the identity

$$\begin{aligned} x(\theta)s(\theta) &= xs + \theta_1(\gamma\mu e - xs)^- + \theta_2(\gamma\mu e - xs)^+ + u(\theta)v(\theta) \\ &= \gamma\mu e + (xs-\gamma\mu e) - \theta_1(xs-\gamma\mu e)^+ - \theta_2(xs-\gamma\mu e)^- + u(\theta)v(\theta) \end{aligned}$$

and (3.22) we deduce the following relation:

$$\begin{aligned} &x(\theta)s(\theta) - \gamma\mu(\theta)e \\ &\quad = \gamma\mu e + (1-\theta_1)(xs-\gamma\mu e)^+ + (1-\theta_2)(xs-\gamma\mu e)^- + u(\theta)v(\theta) - \gamma\mu(\theta)e \\ &\quad = (1-\theta_1)(xs-\gamma\mu e)^+ + (1-\theta_2)(xs-\gamma\mu e)^- \\ &\qquad + \gamma\theta_1\frac{e^T(xs-\gamma\mu e)^+}{n}\,e + \gamma\theta_2\frac{e^T(xs-\gamma\mu e)^-}{n}\,e + u(\theta)v(\theta) - \gamma\frac{u(\theta)^T v(\theta)}{n}\,e\cdot \end{aligned}$$

On the other hand we have

$$\begin{aligned} &\theta_1\frac{e^T(xs-\gamma\mu e)^+}{n} + \theta_2\frac{e^T(xs-\gamma\mu e)^-}{n} \\ &\quad = \theta_1\frac{e^T(xs-\gamma\mu e)}{n} + (\theta_2-\theta_1)\frac{e^T(xs-\gamma\mu e)^-}{n} \\ &\quad \geq (1-\gamma)\mu\theta_1 - (\theta_2-\theta_1)\frac{\alpha\gamma\mu}{\sqrt{n}} = \left(1-\gamma+\frac{\alpha\gamma}{\sqrt{n}}\right)\mu\theta_1 - \frac{\alpha\gamma\mu}{\sqrt{n}}\theta_2 \geq 0. \end{aligned}$$

Therefore we obtain the relation

$$x(\theta)s(\theta) - \gamma\mu(\theta)e \geq (1-\theta_2)(xs-\gamma\mu e)^- + \left(u(\theta)v(\theta) - \gamma\frac{u(\theta)^T v(\theta)}{n}\,e\right)^-,$$

which implies

$$\begin{aligned} \left\| (x(\theta)s(\theta) - \gamma\mu(\theta)e)^- \right\|_2 &\leq (1-\theta_2)\left\| (xs-\gamma\mu e)^- \right\|_2 + \left\| u(\theta)v(\theta) - \gamma\frac{u(\theta)^T v(\theta)}{n}\,e \right\|_2 \\ &\leq (1-\theta_2)\gamma\alpha\mu + \| u(\theta)v(\theta) \|_2 \\ &\leq \left((1-\theta_2)\,\gamma\alpha + \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(n\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{1-\alpha}\right)\right)\mu. \qquad ∎ \end{aligned}$$

Corollary 3.8. *If $0 < \gamma \le 0.5$, then the point $z(\theta)$ defined in* (3.5) *belongs to* $\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ $\forall\,\theta = [\,\theta_1\,;\,\theta_2\,]$ *satisfying*

$$\theta_1 = \frac{t\alpha\gamma\theta_2}{(1-\gamma)\sqrt{n}}, \quad t \in [1, \sqrt{2}\,], \quad 0 < \theta_2 \le \widetilde{\theta}_2 := \frac{(1-\alpha)(1-\gamma)^2}{1+4\sqrt{2}\,\kappa}\,\cdot \tag{3.24}$$

Proof. We first note that under the hypothesis of our corollary we have

$$(1-\gamma)\theta_1 = \frac{t\alpha\gamma\theta_2}{\sqrt{n}} \le \gamma\alpha\theta_2 < \frac{1}{2}, \quad \theta_1 \le \alpha\theta_2\,\cdot$$

From (3.19), $n \ge 2$, and $\alpha(1-\alpha) \le 1/4$ it follows that

$$\begin{aligned}\frac{\mu(\theta)}{\mu} &\ge 1-(1-\gamma)\,\theta_1 - \kappa\left(\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{n(1-\alpha)}\right) > \frac{1}{2} - \kappa\left(\alpha^2\theta_2^2 + \frac{\gamma\alpha^2\theta_2^2}{n(1-\alpha)}\right)\\ &> \frac{1}{2} - \frac{\kappa}{(1+4\sqrt{2}\,\kappa)^2}\left(\alpha^2(1-\alpha)^2 + \frac{\alpha^2(1-\alpha)}{4}\right) > \frac{1}{2} - \frac{\kappa}{8(1+4\sqrt{2}\,\kappa)^2} > 0.\end{aligned}$$

Using Lemmas 3.6 and 3.7 we deduce that

$$\begin{aligned}&\left\|\,(x(\theta)s(\theta)-\gamma\mu(\theta)e)^-\,\right\|_2 - \alpha\gamma\mu(\theta)\\ &\quad\le \left((1-\theta_2)\,\gamma\alpha + \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(n\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{1-\alpha}\right)\right)\mu\\ &\qquad - \gamma\alpha\left(1-(1-\gamma)\,\theta_1 - \kappa\left(\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{n(1-\alpha)}\right)\right)\mu\\ &\quad\le \left(\left(\frac{1}{\sqrt{8}}+2\kappa\right)\left(n\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{1-\alpha}\right) - (\theta_2-(1-\gamma)\,\theta_1)\,\gamma\alpha\right)\mu\\ &\quad= \left(\frac{1+4\sqrt{2}\kappa}{\sqrt{8}}\left(\frac{t^2}{(1-\gamma)^2}+\frac{1}{\gamma(1-\alpha)}\right)\alpha\gamma\theta_2 - \left(1-\frac{t\alpha\gamma}{\sqrt{n}}\right)\right)\gamma\alpha\theta_2\mu.\end{aligned} \tag{3.25}$$

If

$$\theta_2 \le \frac{(1-\gamma)^2(1-\alpha)}{1+4\sqrt{2}\kappa},\ 1 \le t \le \sqrt{2},\ 0 < \gamma \le 0.5,$$

then

$$\begin{aligned}&\frac{1+4\sqrt{2}\kappa}{\sqrt{8}}\left(\frac{t^2}{(1-\gamma)^2}+\frac{1}{\gamma(1-\alpha)}\right)\alpha\gamma\theta_2 - \left(1-\frac{t\alpha\gamma}{\sqrt{n}}\right)\\ &\quad\le \frac{1}{\sqrt{8}}\left(t^2\alpha\gamma(1-\alpha)+\alpha(1-\gamma)^2\right) + \frac{t\alpha\gamma}{\sqrt{n}} - 1\\ &\quad\le \frac{1}{\sqrt{8}} + \alpha\gamma - 1 < \frac{1}{\sqrt{8}} - \frac{1}{2} < 0\,\cdot\end{aligned} \tag{3.26}$$

In deducing the above inequality we have used the fact that

$$\max_{\alpha\in[0,1],\gamma\in[0,\,0.5]} t^2\alpha\gamma(1-\alpha)+\alpha(1-\gamma)^2 = 1\ \forall\,t \in \left[0, \frac{\sqrt{24}+\sqrt{32}}{4}\right], \tag{3.27}$$

which can easily be proved by realizing that the function to be maximized is convex in γ and hence the maximum is attained for $\gamma = 0$ or $\gamma = 0.5$. It turns out that for t in the specified range the maximum is attained for $\gamma = 0$ and $\alpha = 1$. Thus under the hypothesis of our corollary we have $\mu(\theta) > 0$ and $\delta^-_{2,\gamma}(z(\theta)) \leq \alpha$. To complete the proof we invoke Corollary 3.2. ∎

In the remainder of this paper we will use the notation

$$\widetilde{\theta}_2 := \frac{(1-\alpha)(1-\gamma)^2}{1+4\sqrt{2}\,\kappa}, \quad \widetilde{\theta}_1 := \frac{\sqrt{2}\alpha\gamma\widetilde{\theta}_2}{(1-\gamma)\sqrt{n}}, \quad \widetilde{\theta} := \left[\widetilde{\theta}_1\,;\,\widetilde{\theta}_2\right]. \tag{3.28}$$

THEOREM 3.9. *If HLCP is sufficient, then Algorithm* 1 *is well defined, produces a sequence of points* z^k *belonging to the neighborhood* $\widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$, *and*

$$\mu_{k+1} \leq \left(1 - \frac{c(\alpha,\gamma)}{(1+4\sqrt{2}\,\kappa)\sqrt{n}}\right)\mu_k, \quad k = 0, 1, \ldots,$$

where

$$c(\alpha,\gamma) = \frac{\alpha\gamma(1-\alpha)(1-\gamma)^2}{5} \tag{3.29}$$

and κ *is the handicap of the HLCP.*

Proof. The first part of the theorem follows from the definition of Algorithm 1 and Corollary 3.8. Let us denote

$$\phi_1(\theta) = \phi_1(\theta_1,\theta_2) = 1 - \left(1-\gamma+\frac{\alpha\gamma}{\sqrt{n}}\right)\theta_1 + \frac{\gamma\alpha\theta_2}{\sqrt{n}} + \frac{\theta_1^2}{4} + \frac{\gamma\alpha^2\theta_2^2}{4n(1-\alpha)},$$

so that according to (3.18) we have $\mu(\theta) \leq \phi_1(\theta)\mu$.

It is easily verified that

$$\begin{aligned}
\phi_2(\theta_2) &:= \phi_1\left(\frac{t\alpha\gamma\theta_2}{(1-\gamma)\sqrt{n}}, \theta_2\right)\\
&= 1 - \frac{t\left(1-\gamma+\frac{\alpha\gamma}{\sqrt{n}}\right)\alpha\gamma\theta_2}{(1-\gamma)\sqrt{n}} + \frac{\gamma\alpha\theta_2}{\sqrt{n}} + \left(\frac{t^2}{(1-\gamma)^2} + \frac{1}{\gamma(1-\alpha)}\right)\frac{\alpha^2\gamma^2\theta_2^2}{4n}\\
&\leq 1 - \frac{(t-1)\,\alpha\gamma\theta_2}{\sqrt{n}} + \left(\frac{t^2}{(1-\gamma)^2} + \frac{1}{\gamma(1-\alpha)}\right)\frac{\alpha^2\gamma^2\theta_2^2}{4n}\\
&= 1 - \frac{(t-1)\,\alpha\gamma\theta_2}{\sqrt{n}}\left(1 - \left(\frac{t^2}{(1-\gamma)^2} + \frac{1}{\gamma(1-\alpha)}\right)\frac{\alpha\gamma\theta_2}{4(t-1)\sqrt{n}}\right).
\end{aligned}$$

If θ_2 satisfies (3.24), then by taking $t = \sqrt{2}$ and proceeding as in the proof of Corollary 3.8 we obtain

$$\begin{aligned}
\phi_2(\theta_2) &\leq 1 - \frac{\left(\sqrt{2}-1\right)\alpha\gamma\theta_2}{\sqrt{n}}\left(1 - \frac{2\alpha\gamma(1-\alpha)+\alpha(1-\gamma)^2}{4(\sqrt{2}-1)\sqrt{n}}\right)\\
&\leq 1 - \frac{\left(\sqrt{2}-1\right)\alpha\gamma\theta_2}{\sqrt{n}}\left(1 - \frac{1}{4(\sqrt{2}-1)\sqrt{n}}\right)\\
&\leq 1 - \frac{\left(\sqrt{2}-1\right)\alpha\gamma\theta_2}{\sqrt{n}}\left(1 - \frac{1}{4(2-\sqrt{2})}\right) < 1 - \frac{\alpha\gamma\theta_2}{5\sqrt{n}}\,.
\end{aligned}$$

With the notation from (3.28), it follows that

$$\mu(\theta^+) \le \mu(\widetilde{\theta}) \le \left(1 - \frac{c(\alpha,\gamma)}{\left(1+4\sqrt{2}\,\kappa\right)\sqrt{n}}\right)\mu. \qquad \square$$

COROLLARY 3.10. *Under the hypothesis of Theorem* 3.9*, Algorithm* 1 *produces a point* $z \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ *with* $\mu(z) \le \varepsilon$ *in at most* $O((1+\kappa)\sqrt{n}L)$ *iterations, where*

$$L = L_\varepsilon = \log\left(\frac{\mu(z^0)}{\varepsilon}\right). \tag{3.30}$$

3.2. A simplified large update algorithm. Algorithm 1 requires the solution of the two-dimensional optimization problem (3.8). In this section we will show that we can preserve polynomial complexity by using a simple numerical scheme for finding an approximate solution of this problem. We first note that the objective function of (3.8) is a quadratic function in θ. If HLCP is monotone, then $\mu(\theta)$ is convex, but this property does not hold for general sufficient HLCP. For monotone linear complementarity problems, Ai and Zhang [1] proposed a numerical scheme involving just a one-dimensional optimization problem for approximately solving the corresponding two-dimensional optimization problem and implemented this scheme in an algorithm [1, Algorithm 4.3] that proved to be very efficient in practice. It is possible to adapt that algorithm for the case of a $P_*(\kappa)$ HLCP provided the handicap κ is known. Since this is in general not the case, in what follows we will devise a numerical scheme that does not use κ explicitly and has polynomial computational complexity. We know that the choice (3.28) ensures computational complexity, but the expressions in (3.28) depend on κ and the corresponding algorithm would be a short step method with poor practical performance. In contrast the algorithm described below is adaptive and has shown good practical performance on our preliminary tests.

Let us denote

$$\alpha_1 = \frac{e^T(\gamma\mu e - xs)^-}{n\mu},\ \alpha_2 = \frac{e^T(\gamma\mu e - xs)^+}{n\mu},$$
$$\alpha_{11} = \frac{e^T(u^1v^1)}{n\mu},\ \alpha_{12} = \frac{e^T(u^1v^2+u^2v^1)}{2n\mu},\ \alpha_{22} = \frac{e^T(u^2v^2)}{n\mu}.$$

Since

$$u(\theta)v(\theta) = \theta_1^2 u^1v^1 + \theta_1\theta_2\left(u^1v^2+u^2v^1\right) + \theta_2^2u^2v^2$$

we have

$$\psi(\theta) := \frac{\mu(\theta)}{\mu} = 1 + \alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_{11}\theta_1^2 + 2\alpha_{12}\theta_1\theta_2 + \alpha_{22}\theta_2^2. \tag{3.31}$$

Minimizing $\psi(\theta)$ over a line segment in $\mathbb{R}^2$ reduces to minimizing a scalar quadratic function on the interval $[0,1]$, which is trivial. In our algorithm we will minimize $\psi(\theta)$ over a region $\mathcal{E} \subset \mathbb{R}^2$, where $\mathcal{E}$ is either a rectangle of the form

$$R(\overline{\tau}_1,\overline{\tau}_2) = [0,\overline{\tau}_1] \times [0,\overline{\tau}_2] \tag{3.32}$$

with vertices

$$[\,0\,;\,0\,],\quad [\,\overline{\tau}_1\,;\,0\,],\quad [\,\overline{\tau}_1\,;\,\overline{\tau}_2\,],\quad [\,0\,;\,\overline{\tau}_2\,]$$

or a triangle of the form

$$T(\overline{\tau}_2)=\left\{\theta=[\,\theta_1\,;\,\theta_2\,]\ :\ \frac{\alpha\gamma\theta_2}{(1-\gamma)\sqrt{n}}\le\theta_1\le\frac{\sqrt{2}\alpha\gamma\theta_2}{(1-\gamma)\sqrt{n}},\ 0\le\theta_2\le\overline{\tau}_2\right\} \tag{3.33}$$

with vertices

$$[\,0\,;\,0\,],\quad \left[\frac{\sqrt{2}\alpha\gamma\overline{\tau}_2}{(1-\gamma)\sqrt{n}}\,;\,\overline{\tau}_2\right],\quad \left[\frac{\alpha\gamma\overline{\tau}_2}{(1-\gamma)\sqrt{n}}\,;\,\overline{\tau}_2\right].$$

Our next algorithm is based on the observation that minimizing the quadratic function ψ from (3.31) over $\mathcal{E}$ is very inexpensive. If the Hessian of ψ is positive definite and if

$$y^*=-\,(\alpha_1,\alpha_2)\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{12} & \alpha_{22}\end{pmatrix}^{-1}\in\mathcal{E},$$

then the minimizer of ψ over $\mathcal{E}$ is y^*. Otherwise the minimizer of ψ over $\mathcal{E}$ is found by minimizing ψ over the sides of $\mathcal{E}$, which is trivial. The proof of the complexity of the algorithm will be based on the following lemma.

LEMMA 3.11. *Assume that HLCP is $P_*(\kappa)$ and consider the notation form* (3.28). *If $\theta\in T(\rho\,\widetilde{\theta}_2)$ and $0<\rho\le 1$, then $z(\theta)\in\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$. Moreover, $\rho\,\widetilde{\theta}\in T(\rho\,\widetilde{\theta}_2)$ and*

$$\mu(\rho\,\widetilde{\theta})\le\left(1-\frac{\rho\,c(\alpha,\gamma)}{(1+4\sqrt{2}\,\kappa)\,\sqrt{n}}\right)\mu.$$

Proof. The first part of the lemma follows from Corollary 3.8 and the second part of the lemma follows from the proof of Theorem 3.9. ◻

PROCEDURE 1.

Given $z\in\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$, and parameters $\alpha,\gamma,\rho_1,\rho_2\in(0,1)$, with $\gamma\le 0.5$:
  Consider the notation from (3.1)–(3.5), (3.17), (3.32), and (3.33).
  Define $\omega=\sqrt{2}\alpha\gamma/((1-\gamma)\sqrt{n}\,)$.
  Set $\overline{\tau}_2\leftarrow 1$;
  bo (*begin outer iteration*)
    Set $\overline{\tau}_1\leftarrow\overline{\tau}_2$;
    bi (*begin inner iteration*)
      (i1) Compute $y^*=\operatorname{argmin}\{\mu(\theta)\ :\ \theta\in R(\overline{\tau}_1,\overline{\tau}_2)\}$;
      (i2) If $z(y^*)\in\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$, return $(\theta^+,z^+)=(y^*,z(y^*))$ and **terminate**;
      (i3) If $\overline{\tau}_1>\omega\overline{\tau}_2$ set $\overline{\tau}_1\leftarrow\rho_1\overline{\tau}_1$ and go to **bi**;
    ei (*end inner iteration*)
  (o1) Compute $y^*=\operatorname{argmin}\{\mu(\theta)\ :\ \theta\in T(\overline{\tau}_2)\}$;
  (o2) If $z(y^*)\in\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$, return $(\theta^+,z^+)=(y^*,z(y^*))$ and **terminate**;
  (o3) Set $\overline{\tau}_2\leftarrow\rho_2\overline{\tau}_2$ and go to **bo**;
  eo (*end outer iteration*)

We note that $\overline{\tau}_2$ is modified only in the outer iteration. At the beginning of the inner iteration we have $\overline{\tau}_1 = \overline{\tau}_2$ so that one minimizes $\mu(\theta)$ over the square $R(\overline{\tau}_1, \overline{\tau}_2)$. If the corresponding point fails to belong to $\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ the value of $\overline{\tau}_1$ is reduced by a factor of ρ_1 and the procedure is repeated until either we obtain a point in $\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ or $\overline{\tau}_1$ becomes smaller than $\omega\overline{\tau}_2$. In the latter case we switch from minimizing over the rectangle $R(\overline{\tau}_1, \overline{\tau}_2)$ to minimizing over the triangle $T(\overline{\tau}_2)$. This is done in order to ensure that the procedure terminates in a finite number of steps, as shown in the following proposition.

Lemma 3.12. *Procedure* 1 *terminates after at most* $O\left((n\log n)\log(1+\kappa)\right)$ *arithmetic operations, and upon termination we have*

$$\overline{z}^{+} = z(\theta^{+}) \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha), \quad \mu^{+} = \mu(\theta^{+}) < \left(1 - \frac{\rho_1\rho_2 c(\alpha,\gamma)}{\left(1+4\sqrt{2}\,\kappa\right)\sqrt{n}}\right)\mu.$$

Proof. We first analyze the case when Procedure 1 does not terminate at (i2). Then each inner iteration terminates as soon as $\overline{\tau}_1 \le \omega\overline{\tau}_2$, i.e., in $O(|\log\omega|) = O(\log n)$ steps (cf. (i3)). If the outer iteration terminates at (o2) with $\overline{\tau}_2 > \widetilde{\theta}_2$, then $\widetilde{\theta} \in T(\overline{\tau}_2)$, and we have

$$\mu^{+} = \mu(\theta^{+}) \le \mu(\widetilde{\theta}) \le \left(1 - \frac{c(\alpha,\gamma)}{\left(1+4\sqrt{2}\,\kappa\right)\sqrt{n}}\right)\mu. \tag{3.34}$$

Otherwise, according to Lemma 3.11, the outer iteration terminates at (o2) as soon as $\overline{\tau}_2 \le \widetilde{\theta}_2$, i.e., in $O(|\log\widetilde{\theta}_2|) = O(\log(1+\kappa))$ steps. Upon termination we have $\overline{\tau}_2 \le \widetilde{\theta}_2 < \overline{\tau}_2/\rho_2$, which implies $\overline{\tau}_2 = \rho\widetilde{\theta}_2$ for some $\rho \in (\rho_2, 1]$. Hence, in this case,

$$\mu^{+} = \mu(\theta^{+}) \le \mu(\rho\widetilde{\theta}) \le \left(1 - \frac{\rho c(\alpha,\gamma)}{\left(1+4\sqrt{2}\,\kappa\right)\sqrt{n}}\right)\mu < \left(1 - \frac{\rho_2 c(\alpha,\gamma)}{\left(1+4\sqrt{2}\,\kappa\right)\sqrt{n}}\right)\mu.$$

Now let us assume that Procedure 1 terminates at (i2) during outer iteration $\overline{i}$ and inner iteration $\overline{j}$. If $\overline{i} = 1$, then $\overline{\tau}_2 = 1$. If $\overline{i} > 1$, then we must have $\rho_2\widetilde{\theta}_2 < \overline{\tau}_2$ since otherwise Procedure 1 would have terminated at (o2) in the previous outer iteration. If $\overline{j} = 1$, then $\overline{\tau}_1 = \overline{\tau}_2$. Otherwise, if $\overline{j} > 1$, then $\overline{\tau}_1 > \rho_1\omega\overline{\tau}_2$ (cf. (i3)). It follows that for any values of $\overline{i}$ and $\overline{j}$ we must have $\rho_2\widetilde{\theta}_2 < \overline{\tau}_2$ and $\overline{\tau}_1 > \rho_1\omega\overline{\tau}_2$, so that $\rho_1\rho_2\widetilde{\theta}_1 = \rho_1\rho_2\omega\widetilde{\theta}_2 < \rho_1\omega\overline{\tau}_2 < \overline{\tau}_1$. This shows that $\rho_1\rho_2\widetilde{\theta} \in R(\overline{\tau}_1, \overline{\tau}_2)$ and by applying Lemma 3.11 we obtain

$$\mu^{+} = \mu(\theta^{+}) \le \mu(\rho_1\rho_2\widetilde{\theta}) \le \left(1 - \frac{\rho_1\rho_2 c(\alpha,\gamma)}{\left(1+4\sqrt{2}\,\kappa\right)\sqrt{n}}\right)\mu.$$

Each outer iteration requires $O(n)$ arithmetic operations plus at most $O(\log n)$ inner iterations. Each of the inner iterations requires $O(n)$ arithmetic operations. Therefore the total number of arithmetic operations is at most $O((n\log n)\log(1+\kappa))$. ∎

By replacing the two-dimensional optimization problem from Algorithm 1 with the simplified two-dimensional search outlined in Procedure 1 we obtain the following interior point method.

ALGORITHM 2.

Given $\alpha, \gamma, \rho_1, \rho_2 \in (0,1)$, with $\gamma \le 0.5$, and starting point $z^0 \in \widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$:

Set $k \leftarrow 0$;

repeat

Set $z = [\,x\,;\,s\,] \leftarrow z^k = [\,x^k\,;\,s^k\,]$, $\mu \leftarrow \mu_k$;

Compute directions $w^i = [\,u^i\,;\,v^i\,]$, $i = 1, 2$, from (3.1) and (3.2);

Compute (θ^+, z^+) by using Procedure 1;

Set $\theta_k \leftarrow \theta^+$, $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow x^{+T} s^+/n$;

Set $k \leftarrow k+1$.

continue

Using Theorem 3.9 and its proof, together with Lemma 3.12, we can easily prove the following result.

THEOREM 3.13. *If HLCP is sufficient, then Algorithm* 2 *is well defined and produces a sequence of points z^k belonging to the neighborhood $\widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$ such that*

$$\mu_{k+1} \le \left(1 - \frac{\rho_1 \rho_2\, c(\alpha,\gamma)}{\left(1 + 4\sqrt{2}\,\kappa\right)\sqrt{n}}\right)\mu_k, \quad k = 0, 1, \ldots,$$

where $c(\alpha,\gamma)$ is defined in (3.29) *and κ is the handicap of the HLCP.*

Each iteration of Algorithm 2 *requires the solution of two linear systems with the same system matrix, which can be obtained with at most $O(n^3)$ arithmetic operations, and a two-dimensional search procedure that involves at most $O\left((n \log n)\log(1+\kappa)\right)$ arithmetic operations.*

COROLLARY 3.14. *Under the hypothesis of Theorem* 3.13, *Algorithm* 2 *produces a point $z \in \widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$ with $\mu(z) \le \varepsilon$ in at most $O((1+\kappa)\sqrt{n}L)$ iterations, where L is defined in* (3.30).

4. A first order corrector-predictor algorithm. Ai and Zhang [1] proposed a predictor-corrector method using $\widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha/2)$ as the "small neighborhood" and $\widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$ as the "large neighborhood," where $0 < \gamma \le 1/4$ and $0 < \alpha \le 1/2$. If the handicap of the HLCP was known, we could generalize their method for sufficient HLCP by defining appropriate "small" and "large" neighborhoods. (See [19] for a predictor-corrector method using $\mathcal{N}^-_\infty$ neighborhoods.) Since we want to devise an algorithm that is independent of the handicap of the problem, we use the ideas of [18, 14] and construct an analogous corrector-predictor method that requires only one neighborhood, $\widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$. We only assume that $0 < \alpha < 1$ and $0 < \gamma < 0.5$, which allows for very large neighborhoods of the type $\widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$ (with α close to 1 and γ close to zero).

The corrector. The purpose of the corrector step is to improve both centrality and optimality. At a typical iteration of the algorithm we are given a point $z \in \widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$ and we compute the vectors w^1, w^2 by solving the linear systems (3.1) and (3.2). Then we consider the point $z(\theta)$ defined in (3.5). In the corrector step we compute the two-dimensional steplength $\overline{\theta} = (\overline{\theta}_1, \overline{\theta}_2)$ along the directions w^1, w^2 as

$$\begin{aligned} \overline{\theta} = \operatorname*{argmin}_{0 \le \theta_1 \le \theta_2 \le 1} \; & \widetilde{\delta}^-_{2,\gamma}(z(\theta)) \\ \text{subject to } \; & 0 < \mu(\theta) \le (1 - 0.15\theta_1)\mu. \end{aligned} \tag{4.1}$$

According to Corollary 3.2, the corresponding "corrected point" satisfies

$$\overline{z} = z(\overline{\theta}\,) \in \widetilde{\mathcal{N}}^-_{2,\gamma}(\overline{\alpha}\,), \text{ where } \overline{\alpha} = \widetilde{\delta}^-_{2,\gamma}(\overline{z}\,) < \alpha. \tag{4.2}$$

The predictor. A predictor step starts with the point $\overline{z} = [\overline{x}\,;\,\overline{s}\,]$ obtained in the corrector step and computes the affine scaling direction $\overline{w} = [\overline{u}\,;\,\overline{v}\,]$ by solving the linear system

$$\begin{cases} \overline{s}\,\overline{u} + \overline{x}\,\overline{v} &= -\overline{x}\,\overline{s}, \\ Q\overline{u} + R\overline{v} &= 0. \end{cases} \tag{4.3}$$

For any $\xi \in (0,1)$ we define

$$\overline{z}\,(\xi) = [\,\overline{x}\,(\xi)\,;\,\overline{s}\,(\xi)\,] = \overline{z} + \xi\overline{w}. \tag{4.4}$$

We have

$$\overline{x}\,(\xi)\overline{s}\,(\xi) = (1-\xi)\overline{x}\,\overline{s} + \xi^2\overline{u}\,\overline{v}\,,\ \overline{\mu} = \frac{\overline{x}^{\,T}\overline{s}}{n},\ \overline{\mu}\,(\xi) = (1-\xi)\overline{\mu} + \frac{\overline{u}^{\,T}\overline{v}}{n}\,\xi^2. \tag{4.5}$$

The stepsize ξ^+ along the affine scaling direction is obtained as

$$\xi^+ = \operatorname*{argmin}_{\xi\in[0,1]} \overline{\mu}\,(\xi) \tag{4.6}$$
$$\text{subject to } \ \overline{\mu}\,(\xi) > 0 \text{ and } \widetilde{\delta}^{\,-}_{2,\gamma}(\overline{z}\,(\xi)) \le \alpha.$$

As a result of the predictor step, we obtain a point

$$z^+ = \left[\,x^+\,;\,s^+\,\right] := \overline{z}\,(\xi^+). \tag{4.7}$$

According to (4.3) $\overline{s}\,\overline{u} + \overline{x}\,\overline{v} + \overline{x}\,\overline{s} = 0$, so that we can use Proposition 3.1 and the proof of Corollary 3.2 to show that $z^+ \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$. Therefore, a new corrector step can be applied and we obtain the following iterative procedure.

Algorithm 3.

Given parameters $\alpha \in (0,1)$, $\gamma \in (0,\,0.5]$, and starting point $z^0 \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$:
 Set $k \leftarrow 0$;
 repeat
  (corrector step)
  Set $z = [\,x\,;\,s\,] \leftarrow z^k = \left[\,x^k\,;\,s^k\,\right]$, $\mu \leftarrow \mu_k$;
  Compute directions $w^i = \left[\,u^i\,;\,v^i\,\right]$, $i = 1,2$, from (3.1) and (3.2);
  Consider the notation from (3.3) and (3.5);
  Compute steplength $\overline{\theta}$ from (4.1);
  Compute $\overline{z} = [\,\overline{x}\,;\,\overline{s}\,] = z(\overline{\theta}\,)$;
  Set $\overline{\theta}_k \leftarrow \overline{\theta}$, $\overline{z}^{\,k} \leftarrow \overline{z}$, $\overline{\mu}_k \leftarrow \overline{x}^{\,T}\overline{s}\,/n$;
  (predictor step)
  Compute direction $\overline{w} = [\,\overline{u}\,;\,\overline{v}\,]$, by solving (4.3);
  Compute ξ^+ as the solution of (4.6);
  Compute z^+ from (4.7);
  Set $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow \mu^+ = \mu(z^+)$;
  Set $k \leftarrow k+1$.
 continue

4.1. Polynomiality of the first order corrector-predictor algorithm. In the following lemma we obtain an upper bound for the quantity $\overline{\alpha}$ from (4.2).

Lemma 4.1. *If $z \in \widetilde{\mathcal{N}}^{-}_{\gamma}(\alpha)$ with $0 < \alpha < 1$ and $0 < \gamma \le 0.5$, then the point $\overline{z}$ obtained by the corrector step satisfies*

$$\overline{z} = z(\overline{\theta}\,) \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\overline{\alpha}\,),\ \text{where}\ \overline{\alpha} \le (1-\widetilde{\sigma})\,\alpha,$$

with $\widetilde{\sigma} = \widetilde{\theta}_2/7$ and $\widetilde{\theta}_2$ given by (3.28).

Proof. If $\phi_1(\theta) \le 1 - 0.15(1-\gamma)\theta_1$, where ϕ_1 is the function introduced in the proof of Theorem 3.9, then according to (3.18) we have $\mu(\theta) \le (1 - 0.15(1-\gamma)\theta_1)\mu$. Therefore the latter inequality is satisfied if

$$\phi_3(\theta_1,\theta_2) := \frac{\gamma\alpha\theta_2}{\sqrt{n}} + \frac{\theta_1^2}{4} + \frac{\gamma\alpha^2\theta_2^2}{4n(1-\alpha)} - 0.85(1-\gamma)\theta_1 \le 0.$$

For any θ_1, θ_2 satisfying (3.24) we have

$$\begin{aligned}
\phi_3(\theta_1,\theta_2) &= \frac{\gamma\alpha\theta_2}{\sqrt{n}} + \frac{t^2\alpha^2\gamma^2\theta_2^2}{4n(1-\gamma)^2} + \frac{\gamma\alpha^2\theta_2^2}{4n(1-\alpha)} - 0.85\frac{t\alpha\gamma}{\sqrt{n}}\theta_2 \\
&= \left(\left(\frac{t^2}{(1-\gamma)^2} + \frac{1}{\gamma(1-\alpha)}\right)\frac{\gamma\alpha\theta_2}{2\sqrt{n}} - 2(0.85t-1)\right)\frac{\gamma\alpha\theta_2}{2\sqrt{n}} \\
&\le \left(\left(t^2\alpha\gamma(1-\alpha) + \alpha(1-\gamma)^2\right)\frac{1}{2\sqrt{n}} - 2(0.85t-1)\right)\frac{\gamma\alpha\theta_2}{2\sqrt{n}} \\
&\le \left(\frac{1}{2\sqrt{n}} - 2(0.85t-1)\right)\frac{\gamma\alpha\theta_2}{2\sqrt{n}} < 0 \quad \forall n \ge 2, \quad \forall t \in [1.4, \sqrt{2}\,].
\end{aligned}$$

In deducing the inequality above we have used (3.27). In particular it follows that $\mu(\widetilde{\theta}) \le (1 - 0.15(1-\gamma)\widetilde{\theta}_1)\mu$.

Let $0 < \sigma \le \theta_2/7$. Using the techniques used in deducing (3.25) and (3.26) we obtain that for any θ satisfying (3.24) there holds

$$\begin{aligned}
&\left\| (x(\theta)s(\theta) - \gamma\mu(\theta)e)^- \right\|_2 - (1-\sigma)\alpha\gamma\mu(\theta) \\
&\le \left((1-\theta_2)\,\gamma\alpha + \left(\frac{1}{\sqrt{8}} + \kappa\right)\left(n\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{1-\alpha}\right)\right)\mu \\
&\quad -(1-\sigma)\gamma\alpha\left(1 - (1-\gamma)\,\theta_1 - \kappa\left(\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{n(1-\alpha)}\right)\right)\mu \\
&\le \left(\left(\frac{1}{\sqrt{8}} + 2\kappa\right)\left(n\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{1-\alpha}\right) - (\theta_2 - (1-\sigma)\,(1-\gamma)\,\theta_1)\,\gamma\alpha + \sigma\alpha\gamma\right)\mu \\
&< \left(\left(\frac{1}{\sqrt{8}} + 2\kappa\right)\left(n\theta_1^2 + \frac{\gamma\alpha^2\theta_2^2}{1-\alpha}\right) - (\theta_2 - (1-\gamma)\,\theta_1)\,\gamma\alpha + \frac{\alpha\gamma\theta_2}{7}\right)\mu \\
&= \left(\frac{1+4\sqrt{2}\kappa}{\sqrt{8}}\left(\frac{t^2}{(1-\gamma)^2} + \frac{1}{\gamma(1-\alpha)}\right)\alpha\gamma\theta_2 - \left(1 - \frac{t\alpha\gamma}{\sqrt{n}}\right) + \frac{1}{7}\right)\gamma\alpha\theta_2\mu \\
&\le \left(\frac{1}{\sqrt{8}} - 1 + \frac{t\alpha\gamma}{\sqrt{n}} + \frac{1}{7}\right)\gamma\alpha\theta_2\mu < 0.
\end{aligned}$$

By taking $\theta_2 = \widetilde{\theta}_2$ and $\sigma = \widetilde{\sigma} = \widetilde{\theta}_2/7$ it follows that

$$\widetilde{\delta}_{2,\gamma}^-(\overline{z}\,) \le \widetilde{\delta}_{2,\gamma}^-(z(\widetilde{\theta})\,) \le (1-\widetilde{\sigma})\alpha.$$

According to Corollary 3.2, this implies $\overline{z} \in \widetilde{\mathcal{N}}_{2,\gamma}^-((1-\widetilde{\sigma})\alpha)$. ∎

THEOREM 4.2. *If HLCP is sufficient, then Algorithm 3 is well defined and produces a sequence of points z^k belonging to the neighborhood $\widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ such that*

$$\mu_{k+1} < \left(1 - \frac{\overline{c}\,(\alpha,\gamma)}{(1+\kappa)\sqrt{n}}\right)\mu_k, \quad k = 0, 1, \ldots,$$

where

$$\overline{c}\,(\alpha,\gamma) = \frac{(1-\gamma)\sqrt{\gamma\alpha(1-\alpha)}}{10} \tag{4.8}$$

and κ is the handicap of the HLCP.

Proof. From (4.5) we have

$$\overline{x}\,(\xi)\overline{s}\,(\xi) - \gamma\overline{\mu}\,(\xi)e = (1-\xi)(\overline{x}\,\overline{s} - \gamma\overline{\mu}\,e) + \left(\overline{u}\,\overline{v} - \gamma\,\frac{\overline{u}^{\,T}\overline{v}}{n}\,e\right)\xi^2. \tag{4.9}$$

Assume that

$$\xi \le \widetilde{\xi} := \sqrt{\frac{\alpha\gamma\widetilde{\sigma}}{2n(1+\kappa)}}, \text{ where } \widetilde{\sigma} = \frac{\widetilde{\theta}_2}{7}\cdot$$

Since $\gamma \le \frac{1}{2}$, $\widetilde{\sigma} \le \frac{1}{7}$ and $n \ge 2$, this implies

$$\xi \le \sqrt{\frac{\alpha}{28n(1+\kappa)}} < \frac{1}{7} \text{ and } \xi^2 \le \frac{\widetilde{\sigma}}{8(1+\kappa)},$$

so that according to Lemma 3.4 we can write

$$\begin{aligned}
\|(\overline{x}\,(\xi)\overline{s}\,(\xi) - \gamma\overline{\mu}\,(\xi)e)^-\|_2 &\le (1-\xi)\|(\overline{x}\,\overline{s} - \gamma\overline{\mu}\,e)^-\|_2 + \|\overline{u}\,\overline{v} - \gamma\,\frac{\overline{u}^{\,T}\overline{v}}{n}\,e\,\|_2\xi^2 \\
&\le (1-\xi)(1-\widetilde{\sigma})\alpha\gamma\overline{\mu} + \|\overline{u}\,\overline{v}\,\|_2\xi^2 \le (1-\xi)(1-\widetilde{\sigma})\alpha\gamma\overline{\mu} + \left(\frac{1}{\sqrt{8}} + \kappa\right)n\overline{\mu}\,\xi^2 \\
&= (1-\xi)\alpha\gamma\left(1 - \widetilde{\sigma} + \left(\frac{1}{\sqrt{8}} + \kappa\right)\frac{n\xi^2}{(1-\xi)\alpha\gamma}\right)\overline{\mu} \\
&< (1-\xi)\alpha\gamma\left(1 - \widetilde{\sigma} + (1+\kappa)\,\frac{7n\xi^2}{6\alpha\gamma}\right)\overline{\mu} \le (1-\xi)\alpha\gamma\left(1 - \frac{5\widetilde{\sigma}}{12}\right)\overline{\mu}\,,
\end{aligned}$$

$$\begin{aligned}
\overline{\mu}\,(\xi) &= (1-\xi)\overline{\mu} + \frac{\overline{u}^{\,T}\overline{v}}{n}\xi^2 \ge (1-\xi)\overline{\mu} - \frac{\kappa\,\xi^2}{n}\left\|\sqrt{\overline{x}\,\overline{s}}\,\right\|_2^2 = (1-\xi-\kappa\,\xi^2)\overline{\mu} \\
&\ge (1-\xi)\left(1 - \frac{7\kappa\xi^2}{6}\right)\overline{\mu} > (1-\xi)\left(1 - \frac{7\widetilde{\sigma}}{48}\right)\overline{\mu} > (1-\xi)\left(1 - \frac{5\widetilde{\sigma}}{12}\right)\overline{\mu}\,.
\end{aligned}$$

It follows that $\overline{\mu}\,(\xi) > 0$ and $\widetilde{\delta}^{-}_{2,\gamma}(\overline{z}\,(\xi)) < \alpha$ $\forall \xi \in (0, \widetilde{\xi}\,]$. On the other hand, using again Lemma 3.4 and $\widetilde{\xi} < 1/7$, we obtain

$$\overline{\mu}\,(\xi) \le (1-\xi)\overline{\mu} + \frac{\xi^2}{4n}\left\|\sqrt{\overline{x}\,\overline{s}}\,\right\|_2^2 = \left(1 - \frac{\xi}{2}\right)^2\overline{\mu} < \left(1 - \frac{27\xi}{28}\right)\overline{\mu}\ \forall \xi \in (0, \widetilde{\xi}\,].$$

In particular, we have

$$\begin{aligned}\overline{\mu}\,(\xi^+) \le \overline{\mu}\,(\widetilde{\xi}) &< \left(1 - \frac{27}{28}\sqrt{\frac{\alpha\gamma\widetilde{\theta}_2}{14n(1+\kappa)}}\right)\overline{\mu} \\ &= \left(1 - \frac{27(1-\gamma)\sqrt{\gamma\alpha(1-\alpha)}}{28\sqrt{14}(1+\kappa)\sqrt{n}}\sqrt{\frac{1+\kappa}{1+4\sqrt{2}\kappa}}\right)\overline{\mu} \\ &< \left(1 - \frac{27(1-\gamma)\sqrt{\gamma\alpha(1-\alpha)}}{56\sqrt{14}\sqrt[4]{2}(1+\kappa)\sqrt{n}}\right)\overline{\mu} \\ &< \left(1 - \frac{(1-\gamma)\sqrt{\gamma\alpha(1-\alpha)}}{10(1+\kappa)\sqrt{n}}\right)\overline{\mu}\,. \qquad \square\end{aligned}$$

COROLLARY 4.3. *Under the hypothesis of Theorem 4.2, Algorithm 3 produces a point $z \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ with $\mu(z) \le \varepsilon$ in at most $O((1+\kappa)\sqrt{n}L)$ iterations, where L is defined in (3.30).*

4.2. Quadratic convergence. In this subsection we prove that the first order corrector-predictor algorithm is quadratically convergent on problems that are nondegenerate in the sense that HLCP has a strict complementarity solution, i.e., the set

$$\mathcal{F}^{\#} = \{z^* = [\,x^*\,;\,s^*\,] \in \mathcal{F}^* \;:\quad x^* + s^* > 0\} \tag{4.10}$$

is nonempty. Here $\mathcal{F}^*$ denotes the solution set of HLCP defined in (2.5). If $\mathcal{F}^{\#}$ is empty we say that HLCP is degenerate.

THEOREM 4.4. *If HLCP is sufficient and nondegenerate, then the sequence $\{\mu_k\}$ produced by Algorithm 3 is quadratically convergent, i.e., $\mu_{k+1} = O(\mu_k^2)$.*

Proof. From the results on the analyticity of the central path [25, 23] it follows that there is a constant χ independent of k so that the predictor directions computed by Algorithm 3 satisfy

$$\|\,\overline{u}\,\|_2 \le \chi\overline{\mu}\,, \quad \|\,\overline{v}\,\|_2 \le \chi\overline{\mu}\,.$$

Since $\lim_{k\to\infty}\overline{\mu}_k = 0$ we may assume that $\overline{\mu}$ is small enough and let ς be a constant such that

$$\overline{\mu} < \frac{1}{\varsigma}, \quad 0 < \widehat{\xi} := 1 - \varsigma\overline{\mu}\,, \quad \varsigma \ge \frac{3\chi^2}{2\alpha\gamma\widetilde{\sigma}},$$

where $\widetilde{\sigma}$ is defined in Lemma 4.1.

Using (4.5) and the inequalities $\|\overline{u}\,\overline{v}\|_2 \le \|\,\overline{u}\,\|_2\,\|\,\overline{v}\,\|_2$, $\overline{u}^T\overline{v} \le \|\,\overline{u}\,\|_2\,\|\,\overline{v}\,\|_2$ we deduce that for any $\xi \in (0, \widehat{\xi}\,]$ there holds

$$\begin{aligned}\|(\overline{x}\,(\xi)\overline{s}\,(\xi) - \gamma\overline{\mu}\,(\xi)e)^-\|_2 &\le (1-\xi)(1-\widetilde{\sigma})\alpha\gamma\overline{\mu} + \|\overline{u}\,\overline{v}\|_2\xi^2 \\ &\le (1-\xi)(1-\widetilde{\sigma})\alpha\gamma\overline{\mu} + (\chi\overline{\mu}\,\xi)^2 \\ &= (1-\xi)\alpha\gamma\left(1 - \widetilde{\sigma} + \frac{\overline{\mu}\,\chi^2\xi^2}{(1-\xi)\alpha\gamma}\right)\overline{\mu} \\ &< (1-\xi)\alpha\gamma\left(1 - \widetilde{\sigma} + \frac{\chi^2}{\varsigma\alpha\gamma}\right)\overline{\mu} \le (1-\xi)\alpha\gamma\left(1 - \frac{\widetilde{\sigma}}{3}\right)\overline{\mu}\end{aligned}$$

and

$$\begin{aligned}\overline{\mu}\,(\xi) &= (1-\xi)\overline{\mu} + \frac{\overline{u}^{\,T}\overline{v}}{n}\xi^2 \geq (1-\xi)\overline{\mu} - \frac{\|\,\overline{u}\,\|_2\,\|\,\overline{v}\,\|_2\,\xi^2}{n} \\ &\geq (1-\xi)\left(1 - \frac{\chi^2\xi^2\overline{\mu}}{n(1-\xi)}\right)\overline{\mu} > (1-\xi)\left(1-\frac{\chi^2}{n\varsigma}\right)\overline{\mu} \\ &\geq (1-\xi)\left(1-\frac{2\alpha\gamma\widetilde{\sigma}}{3n}\right)\overline{\mu} > (1-\xi)\left(1-\frac{\widetilde{\sigma}}{6}\right)\overline{\mu}\,.\end{aligned}$$

Therefore, $\overline{\mu}\,(\xi) > 0$ and $\widetilde{\delta}^{-}_{2,\gamma}(\overline{z}\,(\xi)) \leq \alpha\ \forall\,\xi \in (0, \widehat{\xi}\,]$. According to (4.6) we have

$$\begin{aligned}\mu^+ = \overline{\mu}\,(\xi^+) \leq \overline{\mu}\,(\widehat{\xi}\,) &= (1-\widehat{\xi}\,)\overline{\mu} + \frac{\overline{u}^{\,T}\overline{v}}{n}\widehat{\xi}^{\,2} < (1-\widehat{\xi}\,)\overline{\mu} + \frac{\|\,\overline{u}\,\|_2\,\|\,\overline{v}\,\|_2}{n} \\ &\leq \left(\varsigma + \frac{\chi^2}{n}\right)\overline{\mu}^{\,2} < \left(\varsigma + \frac{\chi^2}{2}\right)\mu^2. \qquad \square\end{aligned}$$

5. A second order corrector-predictor algorithm. In what follows we will construct and analyze a corrector-predictor algorithm that is superlinearly convergent even for degenerate problems. Since better superlinear convergence results can be obtained when the problem is nondegenerate, we consider these two cases separately. However, we present a unified treatment of the two cases by introducing the parameter

$$\nu := \begin{cases} 1 & \text{for general HLCP (default option),} \\ 0 & \text{if the HLCP is known to be nondegenerate.}\end{cases} \tag{5.1}$$

The corrector step will be the same as in Algorithm 3. Given the point $\overline{z}$ produced by the corrector (see (4.1), (4.2)) we obtain the second order predictor directions $\overline{w}^{\,1} = [\,\overline{u}^{\,1}\,;\,\overline{v}^{\,1}\,]$, $\overline{w}^{\,2} = [\,\overline{u}^{\,2}\,;\,\overline{v}^{\,2}\,]$ by solving the linear systems

$$\begin{cases} \overline{s}\,\overline{u}^{\,1} + \overline{x}\,\overline{v}^{\,1} = -(1+\nu)\overline{x}\,\overline{s}\,, \\ Q\overline{u}^{\,1} + R\overline{v}^{\,1} = 0, \end{cases} \qquad \begin{cases} \overline{s}\,\overline{u}^{\,2} + \overline{x}\,\overline{v}^{\,2} = \nu\overline{x}\,\overline{s} - \overline{u}^{\,1}\overline{v}^{\,1}, \\ Q\overline{u}^{\,2} + R\overline{v}^{\,2} = 0. \end{cases} \tag{5.2}$$

For any $\xi \in (0,1)$ we consider the second order curve

$$\overline{z}\,(\xi) = [\,\overline{x}\,(\xi)\,;\,\overline{s}\,(\xi)\,] = \overline{z} + \xi\overline{w}^{\,1} + \xi^2\overline{w}^{\,2}. \tag{5.3}$$

Along this curve we have

$$\overline{x}\,(\xi)\overline{s}\,(\xi) = (1-\xi)^{1+\nu}\overline{x}\,\overline{s} + \xi^3(\overline{u}^{\,1}\overline{v}^{\,2} + \overline{u}^{\,2}\overline{v}^{\,1}) + \xi^4\overline{u}^{\,2}\overline{v}^{\,2}, \tag{5.4}$$

$$\overline{\mu}\,(\xi) = \mu(\overline{z}\,(\xi)) = (1-\xi)^{1+\nu}\overline{\mu} + \xi^3\frac{\overline{u}^{\,1\,T}\overline{v}^{\,2} + \overline{u}^{\,2\,T}\overline{v}^{\,1}}{n} + \xi^4\frac{\overline{u}^{\,2\,T}\overline{v}^{\,2}}{n}\,. \tag{5.5}$$

The second order predictor uses the stepsize

$$\xi^+ = \operatorname*{argmin}_{\xi\in[0,1]} \overline{\mu}\,(\xi) \quad \text{subject to } \ \overline{z}\,(\xi) \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha) \tag{5.6}$$

and the predicted point is defined by

$$z^+ = [\,x^+\,;\,s^+\,] := \overline{z}\,(\xi^+). \tag{5.7}$$

We note that Proposition 3.1 does not apply for the second order predictor and we do not have an analogue of Corollary 3.2. Therefore we cannot prove that the optimization problems (4.6) and (5.6) are equivalent in this case. Since $z^+ \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ by construction, we obtain the following iterative algorithm.

ALGORITHM 4.
Chose $\nu \in \{0,1\}$ according to (5.1).
Given parameters $\alpha \in (0,1)$, $\gamma \in (0,0.5]$, and starting point $z^0 \in \widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$:
Set $k \leftarrow 0$;
repeat
(corrector step)
Set $z = [\,x\,;\,s\,] \leftarrow z^k = [\,x^k\,;\,s^k\,]$, $\mu \leftarrow \mu_k$;
Compute directions $w^i = [\,u^i\,;\,v^i\,]$, $i = 1,2$, from (3.1) and (3.2);
Consider the notation from (3.3) and (3.5);
Compute steplength $\overline{\theta}$ from (4.1);
Compute $\overline{z} = [\,\overline{x}\,;\,\overline{s}\,] = z(\overline{\theta})$;
Set $\overline{\theta}_k \leftarrow \overline{\theta}$, $\overline{z}^k \leftarrow \overline{z}$, $\overline{\mu}_k \leftarrow \overline{x}^T\overline{s}/n$;
(second order predictor step)
Compute directions $\overline{w}^1 = [\,\overline{u}^1\,;\,\overline{v}^1\,]$, $\overline{w}^2 = [\,\overline{u}^2\,;\,\overline{v}^2\,]$ from (5.2);
Compute ξ^+ as the solution of (5.6);
Compute z^+ from (5.7);
Set $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow \mu^+ = \mu(z^+)$;
Set $k \leftarrow k+1$.
continue

5.1. Polynomiality of the second order corrector-predictor algorithm.

THEOREM 5.1. *If HLCP is sufficient, then Algorithm 4 is well defined and produces a sequence of points z^k belonging to the neighborhood $\widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)$. Moreover, the inequalities*

$$\mu_{k+1} < \left(1 - \frac{\hat{c}\,(\alpha,\gamma)}{(1+4\sqrt{2}\kappa)\sqrt{n}}\right)\mu_k, \quad k = 0,1,\ldots,$$

are satisfied, where

$$\hat{c}\,(\alpha,\gamma) = \frac{\sqrt{(1-\alpha)\gamma}\,\sqrt[3]{\alpha(1-\gamma)^2}}{6} \tag{5.8}$$

and κ is the handicap of the HLCP.

Proof. According to Lemma 4.1 the corrector step produces a point $\overline{z} \in \mathcal{D}((1-\overline{\alpha})\gamma) \subset \mathcal{D}((1-\alpha)\gamma)$ (see (2.9)). Using Lemma 3.4 we deduce that the directions given by the second order predictor satisfy

$$\begin{aligned}
\|\overline{w}^1\|_{\overline{z}} &\le (1+\nu)\sqrt{(1+2\kappa)n\overline{\mu}}\,,\\
\left\|\,\overline{u}^1\overline{v}^1\,\right\|_2 &\le (1+\nu)^2\left(\frac{1}{\sqrt{8}}+\kappa\right)n\overline{\mu}\,,\\
\left\|\nu\sqrt{\overline{x}\,\overline{s}} - \frac{\overline{u}^1\overline{v}^1}{\sqrt{\overline{x}\,\overline{s}}}\right\|_2 &\le \nu\left\|\sqrt{\overline{x}\,\overline{s}}\right\|_2 + \frac{\left\|\,\overline{u}^1\overline{v}^1\,\right\|_2}{\sqrt{(1-\alpha)\gamma\overline{\mu}}}\\
&\le \left(\nu + \sqrt{n}\left(\frac{1}{\sqrt{8}}+\kappa\right)\,\frac{(1+\nu)^2}{\sqrt{(1-\alpha)\gamma}}\right)\,\sqrt{n\overline{\mu}}\\
&\le 2n(1+\nu)^2\left(\frac{1}{\sqrt{8}}+\kappa\right)\sqrt{\frac{\overline{\mu}}{(1-\alpha)\gamma}},
\end{aligned}$$

$$\|\overline{w}^{\,2}\|_{\overline{z}} \le \sqrt{1+2\kappa}\left\| \nu\sqrt{\overline{x}\,\overline{s}} - \frac{\overline{u}^{\,1}\overline{v}^{\,1}}{\sqrt{\overline{x}\,\overline{s}}}\right\|_2 \le 2n(1+\nu)^2\left(\frac{1}{\sqrt{8}}+\kappa\right)\sqrt{\frac{(1+2\kappa)\overline{\mu}}{(1-\alpha)\gamma}},$$

$$\left\|\overline{u}^{\,2}\overline{v}^{\,2}\right\|_2 \le \left(\frac{1}{\sqrt{8}}+\kappa\right)\left\| \nu\sqrt{\overline{x}\,\overline{s}} - \frac{\overline{u}^{\,1}\overline{v}^{\,1}}{\sqrt{\overline{x}\,\overline{s}}}\right\|_2^2 \le 4n^2(1+\nu)^4\left(\frac{1}{\sqrt{8}}+\kappa\right)^3\frac{\overline{\mu}}{(1-\alpha)\gamma},$$

$$\le \frac{(1-\alpha)\gamma}{4\sqrt{2}(1+4\sqrt{2}\kappa)}\left(\frac{(1+\nu)(1+4\sqrt{2}\kappa)\sqrt{n}}{\sqrt{(1-\alpha)\gamma}}\right)^4,$$

$$\begin{aligned}\left\|\overline{u}^{\,1}\overline{v}^{\,2}+\overline{u}^{\,2}\overline{v}^{\,1}\right\|_2 &\le \left\|D\overline{u}^{\,1}\right\|_2\left\|D^{-1}\overline{v}^{\,2}\right\|_2 + \left\|D\overline{u}^{\,2}\right\|_2\left\|D^{-1}\overline{v}^{\,1}\right\|_2 \le \|\overline{w}^{\,1}\|_{\overline{z}}\|\overline{w}^{\,2}\|_{\overline{z}}\\ &\le 2n\sqrt{n}(1+\nu)^3\left(\frac{1}{\sqrt{8}}+\kappa\right)\frac{(1+2\kappa)}{\sqrt{(1-\alpha)\gamma}}\,\overline{\mu}\\ &\le \frac{(1-\alpha)\gamma}{\sqrt{2}(1+4\sqrt{2}\kappa)}\left(\frac{(1+\nu)(1+4\sqrt{2}\kappa)\sqrt{n}}{\sqrt{(1-\alpha)\gamma}}\right)^3.\end{aligned}$$

From (5.4) it follows that

$$\begin{aligned}\overline{x}\,(\xi)\overline{s}\,(\xi) &\ge (1-\xi)^{1+\nu}\overline{x}\,\overline{s} - \xi^3\left\|\overline{u}^{\,1}\overline{v}^{\,2}+\overline{u}^{\,2}\overline{v}^{\,1}\right\|_2 e - \xi^4\left\|\overline{u}^{\,2}\overline{v}^{\,2}\right\|_2 e\\ &\ge (1-\overline{\alpha})\gamma(1-\xi)^{1+\nu}\overline{\mu}\,e - \left(\xi^3\left\|\overline{u}^{\,1}\overline{v}^{\,2}+\overline{u}^{\,2}\overline{v}^{\,1}\right\|_2 + \xi^4\left\|\overline{u}^{\,2}\overline{v}^{\,2}\right\|_2\right)e.\end{aligned} \tag{5.9}$$

Let $\eta \le 1/2$ be a constant to be determined later. If

$$0 < \xi \le \widetilde{\xi_1} := \frac{\eta\sqrt{(1-\alpha)\gamma}}{(1+\nu)(1+4\sqrt{2}\kappa)\sqrt{n}}, \tag{5.10}$$

then

$$\begin{aligned}&\xi^3\left\|\overline{u}^{\,1}\overline{v}^{\,2}+\overline{u}^{\,2}\overline{v}^{\,1}\right\|_2 + \xi^4\left\|\overline{u}^{\,2}\overline{v}^{\,2}\right\|_2\\ &\quad\le \frac{\gamma(1-\alpha)}{2(1+4\sqrt{2}\,\kappa)}\left(\sqrt{2}\eta^3+\frac{\eta^4}{2\sqrt{2}}\right)\overline{\mu} < \frac{\gamma(1-\alpha)\eta^3}{1+4\sqrt{2}\,\kappa}\,\overline{\mu} \le \frac{\gamma(1-\alpha)}{8}\,\overline{\mu}.\end{aligned} \tag{5.11}$$

Since $0<\alpha<1$, $n\ge 2$, $0<\gamma\le 1/2$, we have $\widetilde{\xi_1} < 1/(4(1+\nu))$, which implies

$$(1-\xi)^{1+\nu} > \frac{3}{4} \quad \forall\,\xi\in\left(0,\widetilde{\xi_1}\,\right],\ \nu\in\{0,1\}.$$

Therefore

$$\overline{x}\,(\xi)\overline{s}\,(\xi) > \frac{5\gamma(1-\alpha)\overline{\mu}}{8}\,e > 0 \quad \forall\,\xi\in\left(0,\widetilde{\xi_1}\,\right].$$

By using a continuity argument similar to the one in the proof of Corollary 3.2 we deduce that $\overline{x}\,(\xi)>0$, $\overline{s}\,(\xi)>0\ \forall\,\xi\in\left(0,\widetilde{\xi_1}\,\right]$. Since $Q\overline{x}\,(\xi)+R\overline{s}\,(\xi)=b$ it follows that $\overline{z}\,(\xi)\in\mathcal{F}^0\ \forall\,\xi\in\left(0,\widetilde{\xi_1}\right]$, where $\mathcal{F}^0$ is defined in (2.6). According to (5.4) and (5.5) we have

$$\begin{aligned}\left\|\left(\overline{x}\,(\xi)\overline{s}\,(\xi)-\gamma\overline{\mu}\,(\xi)e\right)^-\right\|_2 &\le (1-\xi)^{1+\nu}\left\|\left(\overline{x}\,\overline{s}-\gamma\overline{\mu}\,e\right)^-\right\|_2 + \xi^3\left\|\overline{u}^{\,1}\overline{v}^{\,2}+\overline{u}^{\,2}\overline{v}^{\,1}\right\|_2\\ &\quad+\xi^4\left\|\overline{u}^{\,2}\overline{v}^{\,2}\right\|_2\\ &\le (1-\xi)^{1+\nu}\overline{\alpha}\,\gamma\overline{\mu} + \xi^3\left\|\overline{u}^{\,1}\overline{v}^{\,2}+\overline{u}^{\,2}\overline{v}^{\,1}\right\|_2 + \xi^4\left\|\overline{u}^{\,2}\overline{v}^{\,2}\right\|_2,\\ \overline{\mu}\,(\xi) &\ge (1-\xi)^{1+\nu}\overline{\mu} - \frac{\xi^3}{\sqrt{n}}\left\|\overline{u}^{\,1}\overline{v}^{\,2}+\overline{u}^{\,2}\overline{v}^{\,1}\right\|_2 - \frac{\xi^4}{\sqrt{n}}\left\|\overline{u}^{\,2}\overline{v}^{\,2}\right\|_2,\end{aligned}$$

and by using Lemma 4.1 and (5.11) we deduce that for any ξ satisfying (5.10) there holds

(5.12)

$$\begin{aligned}
&\left\| (\overline{x}(\xi)\overline{s}(\xi) - \gamma\overline{\mu}(\xi)e)^- \right\|_2 - \alpha\gamma\overline{\mu}(\xi) \\
&\quad \le (1-\xi)^{1+\nu}(\overline{\alpha} - \alpha)\gamma\overline{\mu} + \left(1 + \frac{\alpha\gamma}{\sqrt{n}}\right)\left(\xi^3 \left\| \overline{u}^1\overline{v}^2 + \overline{u}^2\overline{v}^1 \right\|_2 + \xi^4 \left\| \overline{u}^2\overline{v}^2 \right\|_2\right) \\
&\quad < -\frac{\alpha\gamma(1-\alpha)(1-\gamma)^2}{7(1+4\sqrt{2}\,\kappa)}(1-\xi)^{1+\nu}\overline{\mu} + \frac{3}{2}\left(\xi^3 \left\| \overline{u}^1\overline{v}^2 + \overline{u}^2\overline{v}^1 \right\|_2 + \xi^4 \left\| \overline{u}^2\overline{v}^2 \right\|_2\right) \\
&\quad < -\frac{3\alpha\gamma(1-\alpha)(1-\gamma)^2}{28(1+4\sqrt{2}\,\kappa)}\overline{\mu} + \frac{3\gamma(1-\alpha)\eta^3}{2(1+4\sqrt{2}\,\kappa)}\overline{\mu}\,.
\end{aligned}$$

By taking $\eta = \sqrt[3]{\alpha(1-\gamma)^2/14}$, it follows that

$$\overline{z}(\xi) \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha) \quad \forall\, 0 < \xi \le \overline{\xi} := \frac{\sqrt{(1-\alpha)\gamma}\,\sqrt[3]{\alpha(1-\gamma)^2}}{2\sqrt[3]{14}(1+4\sqrt{2}\kappa)\sqrt{n}}\,.$$

Using (5.5) and (5.11) we deduce that

$$\begin{aligned}
\overline{\mu}(\overline{\xi}) &\le (1-\overline{\xi})^{1+\nu}\overline{\mu} + \frac{\overline{\xi}^{\,3}}{\sqrt{n}}\left\| \overline{u}^1\overline{v}^2 + \overline{u}^2\overline{v}^1 \right\|_2 + \frac{\overline{\xi}^{\,4}}{\sqrt{n}}\left\| \overline{u}^2\overline{v}^2 \right\|_2 \\
&\le (1-\overline{\xi})\overline{\mu} + \frac{\gamma(1-\alpha)\eta^3}{(1+4\sqrt{2}\,\kappa)\sqrt{n}}\,\overline{\mu} \\
&= \left(1 - \frac{\eta\sqrt{(1-\alpha)\gamma}}{(1+4\sqrt{2}\kappa)\sqrt{n}}\left(\frac{1}{2} - \sqrt{(1-\alpha)\gamma}\,\eta^2\right)\right)\overline{\mu}\,.
\end{aligned}$$

Since $\eta < \sqrt[3]{1/14}$, it follows that

$$\begin{aligned}
\overline{\mu}(\xi^+) \le \overline{\mu}(\overline{\xi}) &< \left(1 - \frac{\eta\sqrt{(1-\alpha)\gamma}}{(1+4\sqrt{2}\kappa)\sqrt{n}}\left(\frac{1}{2} - \frac{1}{\sqrt{2}\sqrt[3]{576}}\right)\right)\overline{\mu} \\
&< \left(1 - \frac{\sqrt{(1-\alpha)\gamma}\,\sqrt[3]{\alpha(1-\gamma)^2}}{6(1+4\sqrt{2}\kappa)\sqrt{n}}\right)\overline{\mu}\,. \qquad \square
\end{aligned}$$

COROLLARY 5.2. *Under the hypothesis of Theorem* 5.1, *Algorithm* 4 *produces a point* $z \in \widetilde{\mathcal{N}}^{\,-}_{2,\gamma}(\alpha)$ *with* $\mu(z) \le \varepsilon$ *in at most* $O((1+\kappa)\sqrt{n}L)$ *iterations, where* L *is defined in* (3.30).

5.2. Superlinear convergence of the second order corrector-predictor algorithm. In the next theorem we establish the superlinear convergence of Algorithm 4.

THEOREM 5.3. *If HLCP is sufficient, then the sequence* $\{\mu_k\}$ *produced by Algorithm* 4 *is superlinearly convergent with Q-order* $3/(1+\nu)$, *i.e.,*

$$\mu_{k+1} = O(\mu_k^{\frac{3}{1+\nu}}).$$

Proof. Using again the results on the analyticity of the central path [25, 23] it follows that there is a constant $\overline{\chi}$ independent of k so that the predictor directions computed by Algorithm 4 satisfy

$$\left\| \overline{u}^{\,i} \right\|_2 \le \overline{\chi}\,\overline{\mu}^{\,\frac{i}{1+\nu}}, \quad \left\| \overline{v}^{\,i} \right\|_2 \le \overline{\chi}\,\overline{\mu}^{\,\frac{i}{1+\nu}}, \quad i = 1, 2.$$

Since $\lim_{k\to\infty}\overline{\mu}_k = 0$ we may assume that $\overline{\mu}$ is small enough, and let $\overline{\zeta}$ be a constant such that

$$\overline{\mu} < \frac{1}{\overline{\zeta}} < 1, \quad 0 < \widehat{\xi} := 1 - \left(\overline{\zeta}\,\overline{\mu}^{\frac{2-\nu}{1+\nu}}\right)^{\frac{1}{\nu+1}}, \quad \overline{\zeta} \geq \frac{63(1+4\sqrt{2}\,\kappa)\overline{\chi}^2}{2\alpha\gamma(1-\alpha)(1-\gamma)^2}\,.$$

If $0 < \xi < 1$ and $\overline{\mu} < 1$, then

$$\begin{aligned}\xi^3 \left\| \overline{u}^1\overline{v}^2 + \overline{u}^2\overline{v}^1 \right\|_2 + \xi^4 \left\| \overline{u}^2\overline{v}^2 \right\|_2 &< \left\| \overline{u}^1\overline{v}^2 + \overline{u}^2\overline{v}^1 \right\|_2 + \left\| \overline{u}^2\overline{v}^2 \right\|_2 \\ &\leq \left\| \overline{u}^1 \right\|_2 \left\| \overline{v}^2 \right\|_2 + \left\| \overline{u}^2 \right\|_2 \left\| \overline{v}^1 \right\|_2 + \left\| \overline{u}^2 \right\|_2 \left\| \overline{v}^2 \right\|_2 \\ &\leq 2\overline{\chi}^2\overline{\mu}^{\frac{3}{1+\nu}} + \overline{\chi}^2\overline{\mu}^{\frac{4}{1+\nu}} < 3\overline{\chi}^2\overline{\mu}^{\frac{3}{1+\nu}}.\end{aligned}$$

Using (5.9) and (5.12) we deduce that the following inequalities are satisfied for any $\xi \in (0, \widehat{\xi}\,]$:

$$\overline{x}(\xi)\overline{s}(\xi) \geq (1-\overline{\alpha})\gamma(1-\xi)^{1+\nu}\overline{\mu}\,e - 3\overline{\chi}^2\overline{\mu}^{\frac{3}{1+\nu}}e > \left(\gamma(1-\alpha)\overline{\zeta} - 3\overline{\chi}^2\right)\overline{\mu}^{\frac{3}{1+\nu}}e > 0,$$

$$\left\| (\overline{x}(\xi)\overline{s}(\xi) - \gamma\overline{\mu}(\xi)e)^- \right\|_2 - \alpha\gamma\overline{\mu}(\xi) < -\frac{\alpha\gamma(1-\alpha)(1-\gamma)^2}{7(1+4\sqrt{2}\,\kappa)}\overline{\zeta}\,\overline{\mu}^{\frac{3}{1+\nu}} + \frac{9\overline{\chi}^2}{2}\overline{\mu}^{\frac{3}{1+\nu}} \leq 0.$$

As in the proof of Theorem 5.1 this implies that $\overline{z}(\xi) \in \widetilde{\mathcal{N}}^-_{2,\gamma}(\alpha)\ \forall\ \xi \in (0, \widehat{\xi}\,]$. From (5.5) we obtain

$$\overline{\mu}(\widehat{\xi}) \leq (1-\widehat{\xi})^{1+\nu}\overline{\mu} + \frac{\widehat{\xi}^3}{\sqrt{n}}\left\| \overline{u}^1\overline{v}^2 + \overline{u}^2\overline{v}^1 \right\|_2 + \frac{\widehat{\xi}^4}{\sqrt{n}}\left\| \overline{u}^2\overline{v}^2 \right\|_2 \leq \left(\overline{\zeta} + \frac{3\overline{\chi}^2}{\sqrt{2}}\right)\overline{\mu}^{\frac{3}{1+\nu}}.$$

Therefore

$$\mu^+ = \mu(\xi^+) \leq \overline{\mu}(\widehat{\xi}) \leq \left(\overline{\zeta} + \frac{3\overline{\chi}^2}{\sqrt{2}}\right)\overline{\mu}^{\frac{3}{1+\nu}} \leq \left(\overline{\zeta} + \frac{3\overline{\chi}^2}{\sqrt{2}}\right)\mu^{\frac{3}{1+\nu}}. \qquad \square$$

6. Conclusions. In the present paper we have considered several interior point methods for solving sufficient HLCPs in the wide neighborhood of the central path introduced by Ai and Zhang [1] in the context of monontone linear complementarity problems. To our knowledge this is the first time that this wide neighborhood is used in conjunction with sufficient linear complementarity problems. Moreover we use the neighborhood $\mathcal{N}^-_{2,\gamma}(\alpha)$ for any value of $\alpha \in (0,1)$, while in [1] this neighborhood is used only for $\alpha \in (0, 1/2]$. Our algorithms do not use explicitly the handicap κ of the problem, and they can solve any sufficient HLCP problems in at most $O((1+\kappa)\sqrt{n}L)$ iterations, where L is defined in (3.30). This improves the best previously known $O((1+\kappa)\log n\ \sqrt{n}L)$ iteration complexity result for interior point methods acting in a wide neighborhood of the central path that use only a fixed number (independent of n) of backsolves per iteration. Algorithm 1 uses one matrix factorization and two backsolves, plus the solution of the two-dimensional optimization problem (3.6) per iteration. If the handicap κ is known, then a short step variant of Algorithm 1, where θ^+ is set equal to $\widetilde{\theta}$ defined in (3.28), would also have $O((1+\kappa)\sqrt{n}L)$ iteration complexity and would require only one matrix factorization and two backsolves at each iteration. Algorithm 2 is a practical variant of Algorithm 1 where this optimization problem is solved approximately by means of an efficient two-dimensional search procedure requiring at most $O((n\log n)\log(1+\kappa))$ arithmetic operations. For very large values of κ this may dominate the cost of the matrix factorization, which with standards methods for dense matrices is $\Theta(n^3)$ arithmetic operations. In this case the $O((1+\kappa)\log n\,\sqrt{n}L)$ iteration complexity translates into $O((1+\kappa)\log n\; n^{3.5}L)$ computational complexity. Algorithm 2 has a better computational complexity when $\log\kappa = o(n^2)$. The example given in [6] has $\log\kappa = \Omega(n)$. We are not aware of any examples where $\log\kappa = \Omega(n^2)$, although this possibility cannot be excluded. Algorithm 3 uses two matrix factorizations, three backsolves, plus the solutions of the two-dimensional optimization problem (4.1) and the one-dimensional optimization problem (4.6) per iteration. Algorithm 4 uses one additional backsolve. Algorithm 3 is Q-quadratically convergent for nondegenerate problems, while Algorithm 4 is superlinearly convergent with Q order 1.5 for general problems and with Q order 3 for nondegenerate problems. It appears that Algorithm 4 is the first interior point method acting in the wide neighborhood of the central path introduced by Ai and Zhang to achieve superlinear convergence for degenerate problems even in the case of monotone linear complementarity problems. It is possible to construct a practical procedure for solving the two-dimensional optimization problem (4.1) while maintaining the same iteration complexity. Because of space limitations this procedure will be described elsewhere, together with some numerical results. Also, using the ideas from [18, section 6] it is possible to construct efficient line search procedures for solving the one-dimensional optimization problems (4.6) and (5.6) with preservation of the superlinear convergence of the corresponding algorithms.

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