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Multihazard Design: Structural Optimization Approach

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Abstract The objective of multihazard structural engineering is to develop methodologies for achieving designs that are safe and cost-effective under multiple hazards. Optimization is a natural tool for achieving such designs. In general, its aim is to determine a vector of design variables subjected to a given set of constraints, such that an objective function of those variables is minimized. In the particular case of structural design, the design variables may be member sizes; the constraints pertain to structural strength and serviceability (e.g., keeping the load-induced stresses and deflections below specified thresholds); and the objective function is the structure cost or weight. In a multihazard context, the design variables are subjected to the constraints imposed by all the hazards to which the structure is exposed. In this paper, we formulate the multihazard structural design problem in nonlinear programming terms and present a simple illustrative example involving four design variables and two hazards: earthquake and strong winds. Results of our numerical experiments show that interior-point methods are significantly more efficient than classical optimization methods in solving the nonlinear programming problem associated with our illustrative example.

Keywords Structural optimization · Multihazard design · Interior-point method

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1 Introduction

Multihazard design is a branch of structural engineering aimed at developing design practices that achieve superior performance by accounting holistically for all the hazards to which a structure is exposed.

For some types of hazard no more than one hazard may be expected to occur at any one time. This is true, for example, of strong winds and earthquakes, the case to which we confine ourselves in this paper. In current practice structures that may be subjected to wind loads and seismic loads are typically not designed by using a multihazard approach. Rather, they are designed separately for each of the two loads; for each structural member the controlling design then corresponds to the more demanding of the two loads.

It is desirable to achieve design synergies, that is, when performing the design for one of the two hazards it is desirable to take advantage of features inherent in the design for the other hazard, and conversely. This can in principle be accomplished in some cases through ad-hoc, practical engineering judgment. For example, structures subjected to storm surge perform better if the lowest floor offers little resistance to the oncoming water flow. This can be achieved through a design in which the vertical area of the first floor consists strictly of the vertical areas of columns or stilts supporting the upper floors. For wind loading, however, such a design could be uneconomical for structural reasons, and it might be appropriate to design the first floor so that the structure supporting the upper floors consists of a trussed network with small ratio of the solid truss area to the first floor's gross vertical area. This design would help to assure good performance under both storm surge and wind loads.

However ingenious and useful ad-hoc design solutions may be, there is a need for fundamental, science-based approaches to multihazard design. The purpose of this paper is to propose such an approach which, though conceptually simple, brings to bear on the multihazard structural design problem the powerful mathematical apparatus inherent in modern optimization theory. Our proposed approach can be briefly described as follows. Instead of designing a structure so that it performs satisfactorily, or optimally, given that it is subjected to seismic loads, and then repeating the design process independently for wind loads, we propose that the structure be designed so that it performs optimally given that it is subjected to either seismic or wind loads [1].

In the next section we describe the methodology from a mathematical viewpoint. We then present an interesting type of structure that will be used to illustrate our methodology, and use the results of our methodology as applied to that structure to demonstrate the advantages of our approach. The numerical experiments show that interior point methods are significantly more efficient in solving the nonlinear programming problem than classical numerical optimization methods. Although the structure considered in the numerical experiments is relatively simple, we argue that our methodology has the potential for being used, more generally, under a wide variety of hazards, types of structure, and structural engineering constraints. Finally, we present our conclusions and suggestions for further research.

2 Formulating the Multihazard Design Problem as a Nonlinear Programming Problem

In structural optimization we wish to determine a vector $d = (d_1, d_2, \dots, d_n)$ of n design variables such that the structure defined by d satisfies a set of given requirements, and a certain objective function related to the cost of the structure is minimized. This can be achieved by solving a nonlinear programming problem (NLP) of the form

$$\begin{aligned} &\min\ f(d),\\ &\text{s.t. } g(d) \le 0. \end{aligned} \tag{1}$$

Here $f : \mathbb{R}^n \to \mathbb{R}$ represents the objective function and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a vector function defining the constrains. The inequality $g(d) \le 0$ is interpreted componentwise, so that the design variables are subject to m constraints. Some of the constrains are simple bound constraints, imposing lower and upper bounds for the values of the design variables, while other constraints can be more complicated, requiring for example that the maximum stress at all points of the structure does not exceed a given threshold. A vector d satisfying all the constraints is called feasible, meaning that the resulting structure satisfies the given requirements. We denote the set of all feasible design vectors by

$$\mathcal{G} = \{d \in \mathbb{R}^n : g(d) \le 0\}. \tag{2}$$

Solving NLP (1) means selecting from all feasible designs the one for which the objective function attains its minimum value, i.e., finding $\overline{d} \in \mathcal{G}$ such that

$$f(\overline{d}) = \min_{d \in \mathcal{G}} f(d). \tag{3}$$

This optimal design vector is sometimes denoted by

$$\overline{d} = \operatorname*{argmin}_{d \in \mathcal{G}} f(d). \tag{4}$$

Multihazard design is required where the structure may be subjected to p different hazards. Each hazard $i \in \{1, 2, \dots, p\}$ imposes a set of m_i constraints on the design variables. If we denote by $g^i : \mathbb{R}^n \to R^{m_i}$ the vector function defining those constraints, then the optimal design problem with respect to hazard i can be written as

$$\begin{aligned} &\min\ f(d),\\ &\text{s.t. } g^i(d) \le 0. \end{aligned} \tag{5}$$

In many situations, it turns out that the optimal design vector for hazard i,

$$\overline{d^i} = \operatorname*{argmin}_{d \in \mathcal{G}_i} f(d), \qquad \mathcal{G}_i = \{d \in \mathbb{R}^n : g^i(d) \le 0\},$$

is not feasible for hazard j, i.e., $\overline{d^i} \notin \mathcal{G}_j$. The common engineering practice for multihazard design is to obtain separately feasible designs $\widetilde{d^i}$ corresponding to each hazard i, $i = 1, 2, \dots, p$. Those designs are used to construct an envelope $\widetilde{d}$ such that the constraints imposed under all hazards are satisfied. While this approach is robust and has relatively low computational complexity, the resulting design is usually suboptimal, in the sense that the objective function can be further minimized. In this paper we will attempt to optimally solve the multihazard design problem by finding a solution of the following NLP:

$$\begin{aligned} &\min\ f(d), \\ &\text{s.t. } g^1(d) \le 0, \\ &\quad\ g^2(d) \le 0, \\ &\quad\ \vdots \\ &\quad\ g^p(d) \le 0. \end{aligned} \tag{6}$$

The set of feasible design vectors for this problem, $\widehat{\mathcal{G}} = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \cdots \cap \mathcal{G}_p$, is given by the design vectors satisfying the constraints imposed by all hazards. Since for any hazard i $\widehat{\mathcal{G}} \subset \mathcal{G}_i$, its solution $\widehat{d} = \text{argmin}_{d\in\widehat{\mathcal{G}}} f(d)$ satisfies

$$f(\widehat{d}) \ge f(\overline{d^i}), \quad i = 1, 2, \dots, p.$$

This means that while the design $\widehat{d}$ may not be optimal when the hazards are treated separately, $f(\widehat{d})$ represents the minimum possible value of the objective function $f(d)$ over all designs d satisfying the constraints imposed by the p hazards.

Of course, the NLP (6), having more constraints, is more difficult to solve than the corresponding optimization problems that are constrained by just one hazard. However, progress made over the past two decades in the nonlinear programming community, especially the advent of interior point methods (see the excellent review of M. Wright [2]) allows the efficient solution of nonlinear programming problems with a large number of constraints of the form (6). This opens new opportunities in approaching complex multihazard design problems.

3 Case Study

The type of structure we are considering consists of equally spaced steel columns supporting a long, straight horizontal metal pipe filled with water. Computer-controlled mirrors located at various distances from the columns and supported at the ground level reflect solar rays onto the pipe, thereby producing steam used for power generation. Such installations can cover large desert areas. In view of their large numbers of identical elements they are excellent candidates for seeking economical design through structural optimization. Our methodology will be applied to the design of the typical supporting columns.

We assume that the steel columns have hollow elliptical cross sections with constant thickness. A vertical force denoted by V, equal to the weight of the pipe and the water it contains, is applied concentrically at the top of the column. Horizontal forces are applied at the center line of the pipe. They can be due to seismic effects or to wind effects. The seismic forces are denoted by F^e, can act in any direction, and are the same for all directions. The wind forces also act from any direction and are denoted by $F^w \cos\alpha$, where α is the angle between the wind direction and the normal to the longitudinal axis of the pipe. They are largest when normal to the pipe and are assumed to be negligible when parallel to the pipe. A more complex version of our problem that would involve global and local buckling exceeds the scope of our work and would be the object of future research.

4 Setting up the Nonlinear Programming Problem for the Case Study

4.1 Geometry of the Column

For any three numbers a, b, w satisfying the inequalities $0 < w < a \leq b$, we denote by $\mathcal{E}(a, b)$ the ellipse that can be represented in the xy plane by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7}$$

and by $\mathcal{ER}(a, b, w)$ the region situated between the outer ellipse $\mathcal{E}(a, b)$ and the inner ellipse $\mathcal{E}(a - w, b - w)$.

We consider a conical column of height H whose horizontal sections at the top and the bottom are described by the regions $\mathcal{ER}(a, b, w)$ and $\mathcal{ER}(A, B, w)$, respectively. Assume that these regions are similar, in the sense that $a/b = A/B$. Therefore, A can be expressed as

$$A = \frac{aB}{b}. \tag{8}$$

If the height H is given, then the geometry of the column is completely determined by the design vector $d = (a, b, B, w)$, where

- $2a$ is the length of the minor axis of the outer ellipse at the top of the column,
- $2b$ is the length of the major axis of the outer ellipse at the top of the column,
- $2B$ is the length of the major axis of the outer ellipse at the bottom of the column,
- w is the wall thickness of the column.

The first drawing from Fig. 1 represents a vertical section through the column along the major axes of the ellipses. The width w is constant along the column. A horizontal section of the column at distance h from the top of the column is represented by the region $\mathcal{ER}(a(h), b(h), w)$, where

$$a(h) = \left(1 - \frac{h}{H}\right)a + \frac{h}{H}\frac{aB}{b}, \qquad b(h) = \left(1 - \frac{h}{H}\right)b + \frac{h}{H}B. \tag{9}$$

Note that

$$a(0) = a, \qquad a(H) = A, \qquad b(0) = b, \qquad b(H) = B. \tag{10}$$

The second drawing from Fig. 1 represents the horizontal section at the top of the column. In the next section we will use the third drawing from Fig. 1 to deduce the

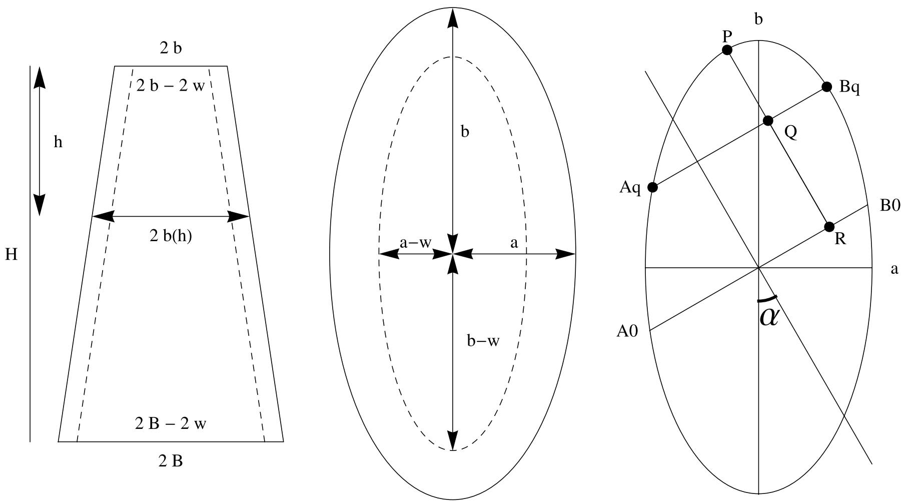

Fig. 1 Column geometry

formula for the area moment of an ellipse with respect of a force whose direction makes an angle α with the major axis of the ellipse.

5 Stresses

In this section, we will find explicit formulas for the stresses.

5.1 Stresses Induced by Gravity Load

At coordinate h from the top of the column, the compression stress due to gravity loading is given by the sum of the vertical concentric load V and the self-weight of the column above the coordinate h, divided by the area area(h) of the region $\mathcal{ER}(a(h), b(h), w)$, i.e.,

$$s_g(a, b, B, w, h) = \frac{V + \gamma \,\mathrm{vol}(h)}{\mathrm{area}(h)},$$

where V was defined earlier, γ is the specific weight of the column material, and

$$\mathrm{area}(h) = \frac{\pi((a+b)(B-b) + bH(a+b-w))wh}{bH},$$

$$\mathrm{vol}(h) = \frac{\pi(2bH(a+b-w) + (a+b)(B-b))wh}{2bH}.$$

Substituting the values above, we obtain the following expression for the stress:

$$s_g(a, b, B, w, h) = \frac{2bHV + 2bH\pi(a+b-w)w\gamma h + (a+b)(B-b)\pi w\gamma h^2}{2\pi(H(a+b-w)b + (a+b)(B-b)h)w}. \quad (11)$$

5.2 Bending Stress

We first derive a formula for the second moment of area of the ellipse $\mathcal{E}(a,b)$ with respect to the axis $\overline{A0, B0}$ that makes an angle α with the major axis of the ellipse (see the third drawing of Fig. 1). We denote by P the point on the ellipse with largest distance to the axis $\overline{A0, B0}$. This distance is equal to the length

$$q_0 = q_0(a,b,\alpha) = \sqrt{0.5(a^2+b^2+(b^2-a^2)\cos(2\alpha))} \tag{12}$$

of the segment $\overline{P, R}$. For any $q \in [0, q_0]$, we consider the segment $\overline{Aq, Bq}$ which is parallel to and at distance q from the segment $\overline{A0, B0}$. The length of $\overline{Aq, Bq}$ is given by

$$l(q) = \frac{2\sqrt{2}ab\sqrt{a^2+b^2-2q^2+(b^2-a^2)\cos(2\alpha)}}{a^2+b^2+(b^2-a^2)\cos(2\alpha)}.$$

With this notation, we obtain the desired moment of area by computing the following integral:

$$\mu_0(a,b,\alpha) = \int_0^{q_0} q^2 l(q)dq = \frac{\pi}{8}ab(a^2+b^2+(b^2-a^2)\cos(2\alpha)). \tag{13}$$

It follows that the second moment of area of the region $\mathcal{ER}(a,b,w)$ about an axis that makes an angle α with the x-axis is given by

$$\mu(a,b,w,\alpha) = \mu_0(a,b,\alpha) - \mu_0(a-w,b-w,\alpha). \tag{14}$$

Finally, we deduce that the bending stress on the column section $\mathcal{ER}(a(h),b(h),w)$, induced by a force F that makes an angle α with the major axis of the ellipse and is applied at a point situated at distance c above the column, can be expressed by the formula

$$s_b(a,b,B,w,F,\alpha,h) = \frac{F(c+h)\,q_0(a(h),b(h),\alpha)}{\mu(a(h),b(h),w,\alpha)}, \tag{15}$$

where $a(h)$ and $b(h)$ are defined by (9).

6 Optimization

In this section, we will give a complete description of the three optimization problems that are studied in this paper.

6.1 Optimization for Earthquake

We assume that the earthquake force has magnitude F^e and can act from any direction. Therefore, it is appropriate to consider a column with circular section, so that, by taking $a=b$, $A=B$, we have only three design variables b, B, w. In this case, the stress induced by the gravity load at distance h from the top of the column becomes

$$s_g^e(b,B,w,h) = \frac{2bHm + 2bH\pi(2b-w)w\gamma h + 2b(B-b)\pi w\gamma h^2}{2\pi(H(2b-w)b+2b(B-b)h)w}. \tag{16}$$

From (13) it follows that when $a = b$ the bending stress does not depend on α and, after some manipulation, we can write it as

$$s_b^e(b, B, w, h) = \frac{8F^e(c+h)H^2\xi}{2\pi((\xi - Hw)^3 + \xi(H^2w^2 - 3H\xi w + 3\xi^2))w},$$
$$\xi = bH + (B-b)h. \tag{17}$$

Therefore the total stress induced by the earthquake and the gravity load at distance h from the top of the column,

$$s^e(b, B, w, h) = s_g^e(b, B, w, h) + s_b^e(b, B, w, h), \tag{18}$$

is a rational function of h of the form

$$s^e(b, B, w, h) = \frac{\sum_{i=0}^{5} \beta_i(b, B, w)h^i}{\sum_{i=0}^{4} \gamma_i(b, B, w)h^i}, \tag{19}$$

with coefficients depending on the design variables. We impose the following simple inequality constraints for the design variables

$$w_{\min} \le w \le a_{\min} \le b \le B \le B_{\max}. \tag{20}$$

Here $w_{\min}$ is a lower bound on the width w and $B_{\max}$ is an upper bound on the major axis of the ellipse at the bottom of the column. The other constraint imposed on the design variables is that the total stress produced by F^e does not exceed a maximum admissible value σ for all $h \in [0, H]$, i.e.,

$$s^e(b, B, w, h) \le \sigma, \quad 0 \le h \le H. \tag{21}$$

Our optimization problem consists of selecting from all possible design variables, that satisfy the constraints (20) and (21), the ones that minimize the weight of the column,

$$\varphi^e(b, B, w) = \gamma\pi H(b + B - w)w. \tag{22}$$

We note that the constraint (21) can be written as $g^e(b, B, w) \le 0$, where

$$g^e(b, B, w) = \max_{h\in[0,H]} s^e(b, B, w, h) - \sigma. \tag{23}$$

With this notation our optimization problem can be written as

$$\begin{aligned} &\min_{b,B,w} \ \varphi^e(b, B, w), \\ &\text{s.t. } g^e(b, B, w) \le 0, \\ &w_{\min} \le w \le a_{\min} \le b \le B \le B_{\max}, \end{aligned} \tag{24}$$

which is of the form (1). We observe that the constraint $g^e(b, B, w) \le 0$ is rather complicated since, for given values of the design variables b, B, w, it involves finding the maximum over the interval $[0, H]$ of the function $h \to s^e(b, B, w, h)$. Formally, the equivalent form (21) involves infinitely many constraints on the design variables b, B, w, one for each value of h. In practice one can replace (21) by a finite number of constraints

$$s^e(b, B, w, h_i) \le \sigma, \quad i = 1, 2, \ldots, k, \tag{25}$$

where $0 \le h_1 \le h_2 \le \cdots \le h_k \le H$ is a finite partition of the interval $[0, H]$. However the constraint (25) is weaker than (21). Remarkably, it is possible to replace the infinite set of constraints (21) by an equivalent finite number of constraints on the design variables. In order to do so we first note that since the denominator in (19) is positive, (21) can be rewritten as

$$-\beta_5(b, B, w)h^5 + \sum_{i=0}^{4}(\sigma\gamma_i(b, B, w) - \beta_i(b, B, w))h^i \ge 0, \quad 0 \le h \le H. \tag{26}$$

Then we could apply a result of [3] that gives necessary and sufficient conditions for the positivity of a polynomial over an interval in terms of a finite number of constraints on the coefficients of the polynomial. This would allow us to phrase our problem as a nonlinear semidefinite programming problem [4–6]. While this approach is very elegant, it is also quite involved, and will not be pursued in the present paper. Instead, as described in the next section, we will apply an algorithm that selects iteratively a partition of the interval $[0, H]$ and uses the solution of the optimization problem constrained by (20) and (25) to obtain design variables a, b and B that satisfy the constraints of (24) and for which the objective value coincides to within two decimals with the optimum value of (24).

6.2 Optimization for Wind

We assume that the wind force, of magnitude F^w, acts along the y-axis, i.e., along the major axis of the ellipse. We consider a column with elliptic region section $\mathcal{ER}(a(h), b(h), w)$. We have four design variables a, b, B, w that must satisfy the simple constraint

$$w_{\min} \le w \le a_{\min} \le a \le b \le B \le B_{\max}. \tag{27}$$

The stress induced by the gravity load, at distance h from the top of the column, is given by (11), while the bending stress is obtained by taking $F = F^w$ and $\alpha = 0$ in (15),

$$s_b^w(a, b, B, w, h) = s_b(a, b, B, w, F^w, 0, h). \tag{28}$$

Therefore the total stress in this case is

$$s^w(a, b, B, w, h) = s_g(a, b, B, w, h) + s_b^w(a, b, B, w, h), \tag{29}$$

This is a rational function in h with coefficients depending on the design variables.

Our objective function, the weight o f the column, is expressed in terms of the design variables as

$$\varphi(a,b,B,w) = \frac{H\pi\gamma(a(b+B)+b(b+B-2w))w}{2b}. \tag{30}$$

In case $a=b$, this reduces to the simpler (polynomial) objective function (22). By denoting

$$g^w(a,b,B,w) = \max_{h\in[0,H]} s^w(a,b,B,w,h) - \sigma, \tag{31}$$

the optimization problem can be written as

$$\begin{aligned} &\min_{a,b,B,w} \ \varphi(a,b,B,w) \\ &\text{s.t. } g^w(a,b,B,w) \le 0, \\ &w_{\min} \le w \le a_{\min} \le a \le b \le B \le B_{\max}. \end{aligned} \tag{32}$$

This has the same form as (24) and all observations made at the end of the previous subsection apply.

6.3 Multihazard Optimization

We now wish to find design variables a, b, B, w so that the stress induced by either wind or earthquake is less than the allowable stress. The constraints imposed by wind are the same as in the previous subsection. However, since the direction of the force F^e of the earthquake is not known the corresponding constraint becomes

$$\begin{aligned} &\overline{s}^e(a,b,B,w,h) = s_g(a,b,B,w,h) + \max_{\alpha\in[0,\pi]} s_b(a,b,B,w,F^e,\alpha,h) \le \sigma, \\ &0 \le h \le H. \end{aligned} \tag{33}$$

If we define

$$\overline{g}^e(a,b,B,w) = \max_{h\in[0,H]} \overline{s}^e(a,b,B,w,h) - \sigma, \tag{34}$$

then our multihazard optimization problem becomes

$$\begin{aligned} &\min_{a,b,B,w} \ \varphi(a,b,B,w) \\ &\text{s.t. } g^w(a,b,B,w) \le 0, \\ &\overline{g}^e(a,b,B,w) \le 0, \\ &w_{\min} \le w \le a_{\min} \le a \le b \le B \le B_{\max}. \end{aligned} \tag{35}$$

This formulation fits into the general framework described in Sect. 2. When comparing (5), (6) with (24), (32) and (35), note that $\overline{\varphi}(b,B,w) = \varphi(a,b,B,w)$ and $g^e(b,B,w) = \overline{g}^e(b,b,B,w)$. The multihazard optimization problem (35) is substantially more difficult to solve than the optimization problems (24), (32), where each hazard is considered separately. However, as we will see in the next section, the multi-hazard optimization problem can be solved efficiently by using modern optimization techniques.

7 Numerical Results

In the numerical results to be presented below we have taken the following values for the constants defining our model

$$H = 3.6 \text{ m}, \qquad c = 0.6 \text{ m}, \qquad V = 50000 \text{ N}, \qquad \gamma = 76820 \text{ N/m}^3,$$
$$F^e = 6000 \text{ N}, \qquad F^w = 8000 \text{ N}, \qquad \sigma = 165 \text{ MPa}. \tag{36}$$

We have chosen the lower bound on the minor semi-axis of the ellipse at the top of the column as $a_{\min} = 4$ cm, and the upper bound one the major semi-axis at the bottom of the column as $B_{\max} = 24$ cm. We have considered four values $w_{\min} = 0.4, 0.3, 0.2, 0.1$ cm for the minimum width of the annulus of the column. While the first two values $w_{\min} = 0.4, 0.3$ cm are reasonable choices, we have also considered smaller values $w_{\min} = 0.2$, 0.1 cm in order to study the behavior of the optimizer. The results will be summarized in Tables 1, 2 and 3 below. While in those tables we give only the rounded values of the optimal design variables a, b, B, w, the computation of the corresponding objective function and stresses is done by using the double precision values of the design variables given by the optimization algorithm.

7.1 Solution of the Optimization for Earthquake Problem

As mentioned in Sect. 6.1, the first constraint in the nonlinear programming problem (24), or equivalently (21), imposes "infinitely many" constraints on the design variables b, B, w, that can be transformed by using the results of [3] into a finite number of (more complicated) constraints on the design variables. Since the results of [3] apply only when the stress is explicitly given as a rational function of h, and since the transformation is rather involved, in this paper we have chosen to replace (21) by a finite number of constraints of the form

$$s^e(b, B, w, h) \le \sigma - \frac{\varepsilon}{2\pi w}, \quad h \in \mathcal{H}, \tag{37}$$

where ε is an appropriately chosen tolerance (in all our experiments we chose $\varepsilon = 0.1$), and $\mathcal{H}$ is a finite family of points from the interval $[0, H]$ that is updated at each iteration of the optimization algorithm. The algorithm relies on the fact that for any given design variables b, B, w, we can compute the maximum in (23) with high accuracy. This is certainly the case given the simple form (19) of $s^e(b, B, w, h)$.

Initially $\mathcal{H}$ contains the endpoints and the midpoint of the interval $[0, H]$,

$$\mathcal{H} = \{0, H/2, H\}. \tag{38}$$

At a typical iteration of our algorithm we find the solution $(b, B, w) = (b, B, w)_{\mathcal{H}}$ of the nonlinear programming problem

$$\begin{aligned} &\min_{b,B,w}\ \varphi^e(b, B, w) \\ &\text{s.t. } s^e(b, B, w, h) \le \sigma - \frac{\varepsilon}{2\pi w}, \quad h \in \mathcal{H}, \\ &w_{\min} \le w \le a_{\min} \le b \le B \le B_{\max}. \end{aligned} \tag{39}$$

Then we use a golden search algorithm to compute, within a tolerance of 0.01ε, a point $h_{\mathcal{H}} \in [0, H]$ at which the maximum of $s^e(b, B, w, h)$ over $[0, H]$ is attained. If

$$s^e(b, B, w, h_{\mathcal{H}}) < \sigma - 0.1\varepsilon, \tag{40}$$

then we return $(b, B, w) = (b, B, w)_{\mathcal{H}}$ as t he optimal solution of our problem. Otherwise we add the point $h_{\mathcal{H}}$ to the set $\mathcal{H}$,

$$\mathcal{H} = \mathcal{H} \cup \{h_{\mathcal{H}}\}, \tag{41}$$

and start a new iteration.

The numerical results obtained with this algorithm are summarized in Table 1, where we list for each value of the lower bound $w_{\min}$, the value of the optimal design variables b, B, w, and the corresponding weight of the column. We also list the maximum stress caused by the wind force (msW), the maximum stress caused by the earthquake force (msQ) and the heights at which they are attained (hW and hQ, correspondingly), i.e.,

$$hW = \operatorname*{argmax}_{h\in[O,H]} s^w(a, b, B, w, h), \qquad msW = s^w(b, b, B, w, hW),$$

$$hQ = \operatorname*{argmax}_{h\in[O,H]} s^e(b, B, w, h), \qquad msQ = s^e(b, B, w, hW). \tag{42}$$

In the last column of the table we list the number of iterations, it, used by algorithm to obtain the optimal design variables. We note that in all four cases we have $msQ = 165$ MPa, as expected. However the maximum stress caused by the wind force (msW) exceeds the allowable limit by about 25%. We also note that the number of iterations is relatively small and is relatively independent of the choice of $w_{\min}$. The

Table 1 Optimization for earthquake ($a_{\min} = 4$ cm, $B_{\max} = 24$ cm)

$w_{\min}$ (cm)	b (cm)	B (cm)	w (cm)	weight (kg)	msW (MPa)	hW (cm)	msQ (MPa)	hQ (cm)	it
0.4	6.126	12.376	0.400	64.15	212.7	188.5	165.0	179.9	10
0.3	7.082	14.280	0.300	55.98	211.7	190.6	165.0	180.6	10
0.2	8.798	17.606	0.200	46.43	210.0	192.5	165.0	180.1	10
0.1	13.411	24.000	0.107	35.45	208.1	267.9	165.0	248.3	9

lower bound $w_{\min}$ is attained in the first three cases, while the bound $a_{\min}$ is never attained, and $B_{\max}$ is attained only in the last case. In fact, as we remarked earlier, we added this case, just for studying the behavior or the optimizer.

7.2 Solution of the Optimization for Wind Problem

The optimization algorithm is similar to the one described in the previous subsection. However now we have four design variables a, b, B, w. The set $\mathcal{H}$ is initially defined as in (38) and at each iteration we compute the solution $(a, b, B, w) = (a, b, B, w)_{\mathcal{H}}$ of the nonlinear optimization problem

$$\begin{aligned} &\min_{a,b,B,w} \ \varphi(a, b, B, w) \\ &\text{s.t. } s^w(a, b, B, w, h) \le \sigma - \frac{\varepsilon}{2\pi w}, \quad h \in \mathcal{H}, \\ &w_{\min} \le w \le a_{\min} \le a \le b \le B \le B_{\max}, \end{aligned} \tag{43}$$

where $\varphi(a, b, B, w)$ is the weight of the column (see (30) and (29)) $s^w(a, b, B, w, h)$ is the stress produced by the wind force at height h (see (29)). Then we compute (within a tolerance of 0.01ε) the height $h_{\mathcal{H}}$ at which this stress attains its maximum

$$h_{\mathcal{H}} = \operatorname*{argmax}_{h\in[O,H]} s^w(a, b, B, w, h), \tag{44}$$

and return $(a, b, B, w) = (a, b, B, w)_{\mathcal{H}}$ as the solution of the optimal design problem if

$$s^w(a, b, B, w, h_{\mathcal{H}}) < \sigma - 0.1\varepsilon. \tag{45}$$

If the above inequality is violated then we update $\mathcal{H}$ as in (41) and repeat the procedure. The results obtained by this algorithm are presented in Table 2. The entries of the table have the same meaning as those in Table 1, with the observation that now

$$hW = \operatorname*{argmax}_{h\in[O,H]} s^w(a, b, B, w, h), \qquad msW = s^w(a, b, B, w, hW), \tag{46}$$

$$hQ = \operatorname*{argmax}_{h\in[O,H]} \bar{s}^e(a, b, B, w, h), \qquad msQ = \bar{s}^e(a, b, B, w, hQ), \tag{47}$$

Table 2 Optimization for wind ($a_{\min} = 4$ cm, $B_{\max} = 24$ cm)

$w_{\min}$ (cm)	a (cm)	b (cm)	B (cm)	w (cm)	weight (kg)	msW (MPa)	hW (cm)	msQ (MPa)	hQ (cm)	it
0.4	4.000	9.251	18.626	0.400	69.33	165.0	187.6	193.3	186.1	8
0.3	4.000	11.199	22.729	0.300	60.39	165.0	181.5	211.6	184.8	8
0.2	10.124	10.362	20.000	0.200	52.83	165.0	212.9	131.4	198.9	8
0.1	13.171	13.171	24.000	0.136	44.75	165.0	251.1	131.0	232.2	8

where $s^w(a,b,B,w,h)$ and $\overline{s}^e(a,b,B,w,h)$ are given by (29) and (33).

We note that in all cases the maximum stress produced by the wind is equal to the maximum allowable stress of 165 MPa. For $w_{\min} = 0.4, 0.3$ cm the maximum stress produced by the earthquake exceeds this threshold. This is not the case for $w_{\min} = 0.2, 0.1$ cm. This is due to the fact that the optimal values of a and b are almost equal in those cases (for $w_{\min} = 0.1$ cm they differ only in the fourth decimal). As we mentioned earlier, setting the lower bound $w_{\min} = 0.1$ in is not realistic for structural reasons; we have considered this case in order to monitor the behavior of the optimization algorithm. We also note that for $w_{\min} = 0.4, 0.3$ cm, the optimum a attains its lower bound $a_{\min} = 4$ cm, while for $w_{\min} = 0.1$ cm the optimum B attains its upper bound $B_{\max} = 24$ cm. Finally we remark that the optimum w attains the lower bound $w_{\min}$ for $w_{\min} = 0.4, 0.3, 0.2$ cm, but is slightly above it for $w_{\min} = 0.1$ cm.

While this behavior is to some extent expected, our numerical results show the importance of correctly setting the bounds on the design variables. The conclusion is that the bounds on the design variables should be set in accordance with the requirements of the given application. A good optimization algorithm will find an optimal solution satisfying the bounds.

7.3 Solution of the Multihazard Optimization Problem

There are slight differences between the optimization algorithm for solving (35) and the algorithms presented in the previous two subsections. In order to increase the robustness of the algorithm we initialize the set $\mathcal{H}$ as

$$\mathcal{H} = \{0, H/2, H, \text{rand}\}, \tag{48}$$

where rand is a randomly chosen number from the interval $[0, H]$. At a typical iteration we compute the solution $(a,b,B,w) = (a,b,B,w)_{\mathcal{H}}$ of the nonlinear optimization problem

$$\begin{aligned}
&\min_{a,b,B,w} \ \varphi(a,b,B,w)\\
&\text{s.t. } s^w(a,b,B,w,h) \le \sigma - \frac{\varepsilon}{2\pi w}, \quad h \in \mathcal{H},\\
&\overline{s}^e(a,b,B,w,h) \le \sigma - \frac{\varepsilon}{2\pi w}, \quad h \in \mathcal{H},\\
&w_{\min} \le w \le a_{\min} \le a \le b \le B \le B_{\max},
\end{aligned} \tag{49}$$

where $s^w(a,b,B,w,h)$ and $\overline{s}^e(a,b,B,w,h)$ are the stresses given by (29) and (33). We then compute the quantities hW, msW, hQ, msQ defined by (44)–(47). If both msW and msQ are less than or equal to σ, we return $(a,b,B,w) = (a,b,B,w)_{\mathcal{H}}$ as the optimal solution of the multihazard optimization problem. Otherwise we update $\mathcal{H}$ by the following procedure:

$$\begin{aligned}
&\text{If } msW > \sigma - 0.1\varepsilon, \quad \text{then } \mathcal{H} = \mathcal{H} \cup \{\chi(hW)\};\\
&\text{If } msQ > \sigma - 0.1\varepsilon, \quad \text{then } \mathcal{H} = \mathcal{H} \cup \{\chi(hQ)\},
\end{aligned} \tag{50}$$

Table 3 Multihazard optimization ($a_{\min} = 4$ cm, $B_{\max} = 24$ cm)

$w_{\min}$ (cm)	a (cm)	b (cm)	B (cm)	w (cm)	weight (kg)	msW (MPa)	hW (cm)	msQ (MPa)	hQ (cm)	it
0.4	4.916	8.370	17.039	0.400	70.05	165.0	180.8	165.0	177.0	7
0.3	5.639	9.705	19.758	0.300	61.10	165.0	179.9	165.0	177.4	8
0.2	10.124	10.361	20.000	0.200	52.83	165.0	212.8	131.4	198.8	10
0.1	13.169	13.169	24.000	0.136	44.75	165.0	251.0	131.0	232.1	6

where $\chi(h) = h$ if the distance between the point h and the set $\mathcal{H}$ is greater than 0.01ε, and $\chi(h)$ is a random number chosen from the interval $[0, H]$ otherwise. The numerical results are listed in Table 3.

For $w_{\min} = 0.4, 0.3$ cm the maximum stress produced by the earthquake is the same as the maximum stress produced by the wind, both being equal to the maximal allowable stress σ. For $w_{\min} = 0.2, 0.1$ cm we obtain, within two decimals, the same results as in Sect. 7.2. Note that in computing the weight we have used the values of a, b, B, w in double precision, as given by the optimization algorithm, while in Tables 2 and 3 we have displayed approximations of those values.

Nevertheless, for $w_{\min} = 0.2, 0.1$ cm the optimal multihazard design is basically the same as the optimal design where only one hazard (wind) is considered. In fact we added these two unrealistic values for $w_{\min}$ just in order to show that this phenomenon can actually happen. It will always happen if one of the hazards is much stronger than the others.

7.4 Computational Complexity

Most of the computer time used for obtaining the numerical results presented above was spent in solving the nonlinear programming problems (39), (43) and (49) at each iteration of our solution algorithm for the earthquake (24), wind (32), and multihazard (35) optimization problems. Note that the number of constraints of the respective nonlinear programming problem increases at each iteration. Since, as seen from Tables 1, 2 and 3, the number of iterations varies from 6 to 10, the number of constraints remains relatively small so that the corresponding nonlinear programming problems can be solved by any robust nonlinear programming solver. The results from Tables 1, 2 and 3 were obtained by using the *Mathematica* function `NMinimize`. However, for structural optimization problems involving a larger number of design variables, the use of more efficient nonlinear programming solvers is essential. We believe that interior-point method based solvers [2] are ideally suited for large scale structural optimization problems. Even on the small scale structural optimization problems considered in this paper, the use of interior-point methods has led to substantial speedup. Thus, by using instead of `NMinimize` the *Mathematica* function `FindMinimum` with the option `Method -> "InteriorPoint"` we have reduced the computer time by a factor of more than 40. Further speedup was obtained by using the function `KnitroMinimize` from the commercial package *KNITRO for Mathematica 5.2* from Ziena Optimization Inc. (http://www.ziena.com/). We note that with both

Table 4 Timings for $a_{min} = 4$ cm, $B_{max} = 24$ cm

w_{min} (in)	`NMinimize` (sec)	`FindMinimum` (sec)	`FindMinimum` (speedup)	`KnitroMinimize` (sec)	`KnitroMinimize` (speedup)
0.4	427.14	10.14	42.12	7.20	59.30
0.3	500.73	12.14	41.24	8.51	58.80
0.2	600.73	12.35	48.60	8.26	72.67
0.1	220.93	9.90	22.30	5.17	42.71

`FindMinimum` and `KnitroMinimize` we have used the trivial starting point

$$a = 0.4B_{max}, \qquad b = 0.5B_{max}, \qquad B = 0.9B_{max}, \qquad w = 1.5w_{min} \tag{51}$$

for all nonlinear programming problems. We could have obtained additional speedup by using starting points depending on the solution of the corresponding nonlinear programming problem at the previous iteration. We present the computer time required by the three approaches presented above in Table 4. We have used the *Mathematica* function `Timing` to obtain the total CPU time in seconds required for the solution of the nonlinear programming problems (24), (32) and (35).

8 Conclusions

In this paper we have demonstrated, on a simple example, the potential of using nonlinear optimization techniques for multihazard design. The design variables determined by our approach lead to minimum cost structures that are able to sustain the loads generated by the different hazards under consideration. Recent progress in nonlinear optimization algorithms makes it possible to efficiently solve the resulting nonlinear optimization problems. In particular, the use of interior point methods seems to offer the possibility of using the nonlinear optimization approach on large scale problems. Since for such problems it is impossible to obtain closed form expressions for the loads generated by the different hazards, the techniques described in the present paper will have to modified be accordingly. We are working on an iterative method for solving large scale multihazard design problems, where at each iteration we formulate a nonlinear optimization problem from the information obtained by running numerical simulations of the corresponding structural response.

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