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Corrector-predictor methods for sufficient linear complementarity problems

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Abstract We present a new corrector-predictor method for solving sufficient linear complementarity problems for which a sufficiently centered feasible starting point is available. In contrast with its predictor-corrector counterpart proposed by Miao, the method does not depend on the handicap κ of the problem. The method has $O((1+\kappa)\sqrt{n}L)$-iteration complexity, the same as Miao's method, but our error estimates are sightly better. The algorithm is quadratically convergent for problems having a strictly complementary solution. We also present a family of infeasible higher order corrector-predictor methods that are superlinearly convergent even in the absence of strict complementarity. The algorithms of this class are globally convergent for general positive starting points. They have $O((1+\kappa)\sqrt{n}L)$-iteration complexity for feasible, or "almost feasible", starting points and $O((1+\kappa)^2 nL)$-iteration complexity for "sufficiently large" infeasible starting points.

Keywords Linear complementarity · Interior-point · Superlinear convergence

1 Introduction

In this paper we present a family of corrector-predictor interior point methods for solving sufficient linear complementarity problems. The methods can be seen as variants of the Mizuno-Todd-Ye (MTY) predictor-corrector method. The latter method was proposed in [11] and it was the first interior-point method having both

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polynomial-time complexity and superlinear convergence. More precisely, MTY has $O(\sqrt{n}L)$ iteration complexity for general linear programming problems (LP) of dimension n with integer data of bit-length L (cf. [11]) and it is quadratically convergent (cf. [9, 21]). MTY was generalized to monotone linear complementarity problems (LCP) in [6] and the resulting algorithm was proved to have $O(\sqrt{n}L)$ iteration complexity under general conditions, and superlinear convergence under the assumption that the LCP has a strictly complementary solution and the iteration sequence converges. We mention that according to [2] the latter assumption always holds. We also mention that the former assumption is not restrictive, since according to [12] a large class of interior point methods, which contains MTY, cannot be superlinearly convergent for LCPs that do not have a strictly complementary solution (i.e. are degenerate). Ye and Anstreicher [20] proved that MTY converges quadratically assuming only that the LCP is nondegenerate.

The MTY predictor-corrector algorithm was extended for $P_*(\kappa)$ LCPs in 1995 by Miao [10]. The class of all $P_*(\kappa)$ matrices is called P_*. This class was extensively studied in the 1991 monograph of Kojima et al. [7], where it provided a unified framework for studying interior point methods. In 1996 Väliaho [19] showed that the class of P_* matrices coincides with the class of sufficient matrices introduced in 1989 by Cottle et al. [4] (see also [5]). Thus a matrix is sufficient if it is a $P_*(\kappa)$-matrix for some $\kappa \geq 0$. The smallest such κ is called the handicap of the sufficient matrix. Miao's algorithm has $O((1+\kappa)\sqrt{n}L)$ iteration complexity if the LCP is $P_*(\kappa)$, and is quadratically convergent if the $(P_*(\kappa))$ LCP is nondegenerate. The handicap κ is explicitly used in the construction of Miao's algorithm. Since the handicap of a sufficient matrix is sometimes very difficult to compute, Miao's algorithm cannot be used for general sufficient LCPs.

The first order corrector-predictor algorithm to be presented in this paper does not depend on the handicap of the matrix so that it can be applied for any sufficient LCP. It is quadratically convergent for any nondegenerate sufficient LCP and has the same computational complexity as Miao's algorithm for $P_*(\kappa)$ LCPs. However our estimates are slightly better than those from [10]. In the special case of linear programming our estimates are better than those obtained in the original paper of Mizuno, Todd and Ye [11].

Using higher order predictors we develop a class of corrector-predictor methods that are superlinearly convergent even for degenerate problems. Our computational complexity and superlinear convergence results are similar to the corresponding results for a family of high-order predictor-corrector methods proposed by Stoer, Wechs and Mizuno [16–18], but in contrast with those methods our algorithms do not depend on the handicap of the LCP. We prove that our algorithms are globally convergent for arbitrary positive starting points and have $O((1+\kappa)\sqrt{n}L)$-iteration complexity for feasible, or "almost feasible", starting points and $O((1+\kappa)^2 nL)$-iteration complexity for "sufficiently large" infeasible starting points. The m'th order algorithms require two matrix factorizations and m backsolves per iterations. Their Q-order of convergence is at least $m+1$ for nondegenerate problems and $(m+1)/2$ for degenerate problems.

We present several numerical experiments illustrating the theoretical properties of our algorithms.

Conventions We denote by $\mathbb{N}$ the set of all nonnegative integers, and $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$ denote the set of real, nonnegative real, and positive real numbers respectively. Given a vector x, the corresponding upper case symbol X denotes the diagonal matrix X defined by the vector x. The symbol e represents the vector of all ones with appropriate dimension.

We denote component-wise operations on vectors by the usual notations for real numbers. Thus, given two vectors u and v of the same dimension, uv, u/v, etc. will denote the vectors with components $u_i v_i$, u_i/v_i, etc. This notation is consistent as long as component-wise operations always have precedence in relation to matrix operations. Note that $uv \equiv Uv$ and if A is a matrix, then $Auv \equiv AUv$, but in general $A(uv) \neq (Au)v$. Also, if f is a scalar function and v is a vector, then $f(v)$ denotes the vector with components $f(v_i)$. For example, if $v \in \mathbb{R}^n_+$ and $\lambda \in \mathbb{R}$, then $\sqrt{v}$ denotes the vector with components $\sqrt{v_i}$, and $\lambda - v$ denotes the vector with components $\lambda - v_i$. Traditionally the vector $\lambda - v$ is written as $\lambda e - v$. If $\|.\|$ is a vector norm on $\mathbb{R}^n$ and A is a matrix, then the operator norm induced by $\|.\|$ is defined by $\|A\| := \max\{\|Ax\| : \|x\| = 1\}$. As a particular case we note that if U is the diagonal matrix defined by the vector u, then $\|U\|_2 = \|u\|_\infty$.

If x and s are vectors in $\mathbb{R}^n$ and τ is a scalar in $\mathbb{R}$, then the vector $z \in \mathbb{R}^{2n}$ obtained by concatenating x and s is denoted by $z = \lceil x, s \rfloor = [x^T, s^T]^T$, the mean value of xs is denoted by $\mu(z) := (x^T s)/n$, and $\lceil x, s, \tau \rfloor := [x^T, s^T, \tau]^T$.

2 The sufficient homogeneous linear complementarity problem and its central path

Given two matrices Q and R in $\mathbb{R}^{n\times n}$, and a vector b in $\mathbb{R}^n$, the horizontal linear complementarity problem (HLCP) consists in finding a pair of vectors $z = \lceil x, s \rfloor$ such that

$$\begin{aligned} xs &= 0, \\ Qx + Rs &= b, \\ x, s &\geq 0. \end{aligned} \tag{1}$$

The standard (monotone) linear complementarity problem (SLCP or simply LCP) corresponds to the case where $R = -I$, and Q is positive semidefinite. Let $\kappa \geq 0$ be a given constant. We say that (1) is a $P_*(\kappa)$ HLCP if

$$Qu + Rv = 0 \text{ implies } (1+4\kappa) \sum_{i\in\mathcal{I}^+} u_i v_i + \sum_{i\in\mathcal{I}^-} u_i v_i \geq 0, \quad \text{for any } u, v \in \mathbb{R}^n, \tag{2}$$

where $\mathcal{I}^+ = \{i : u_i v_i > 0\}$ and $\mathcal{I}^- = \{i : u_i v_i < 0\}$. If the above condition is satisfied, then we say that (Q, R) is a $P_*(\kappa)$ pair and write $(Q, R) \in P_*(\kappa)$. In the case $R = -I$, $(Q, -I)$ is a $P_*(\kappa)$ pair if and only if Q is a $P_*(\kappa)$ matrix, that is,

$$(1+4\kappa) \sum_{i\in\hat{\mathcal{I}}^+} x_i [Qx]_i + \sum_{i\in\hat{\mathcal{I}}^-} x_i [Qx]_i \geq 0, \quad \forall x \in \mathbb{R}^n, \tag{3}$$

where $\hat{\mathcal{I}}^+ = \{i : x_i[Qx]_i > 0\}$ and $\hat{\mathcal{I}}^- = \{i : x_i[Qx]_i < 0\}$. Problem (1) is then called a $P_*(\kappa)$ LCP and it is extensively discussed in [7]. If (Q, R) belongs to the class

$$P_* = \bigcup_{\kappa \geq 0} P_*(\kappa),$$

then we say that (Q, R) is a P_* pair and (1) is a P_* HLCP.

As mentioned in the introduction, the class of sufficient matrices was defined by Cottle et al. in [4]. The appropriate generalization to sufficient pair [17, 18] is in terms of the null space of the matrix $[Q\ R] \in \mathbb{R}^{n\times 2n}$

$$\Phi := \mathcal{N}([Q \quad R]) = \{\lceil u, v \rfloor : Qu + Rv = 0\} \tag{4}$$

and its orthogonal space

$$\Phi^\perp = \{\lceil u, v \rfloor : u = Q^T x, v = R^T x, \text{ for some } x \in \mathbb{R}^n\}. \tag{5}$$

The pair (Q, R) is called column sufficient if

$$\lceil u, v \rfloor \in \Phi, \quad uv \leq 0 \text{ implies } uv = 0,$$

and row sufficient if

$$\lceil u, v \rfloor \in \Phi^\perp, \quad uv \geq 0 \text{ implies } uv = 0.$$

(Q, R) is a sufficient pair if it is both column and row sufficient. All main results about row and column sufficient matrices in [5] can be extended to row and column sufficient pairs (see, for example, [15]). For example, we can prove that (Q, R) is a sufficient pair if and only if for any b, the HLCP (1) has a convex (perhaps empty) solution set and every KKT point of

$$\begin{array}{ll} \min & x^T s \\ \text{s.t.} & Qx + Rs = b, \\ & x, s \geq 0 \end{array}$$

is a solution of (1).

Väliaho's result [19] states that a matrix is sufficient if and only if it is a $P_*(\kappa)$ matrix for some $\kappa \geq 0$. The result can be extended to sufficient pairs using the equivalence results from [1] (see also [15]): (Q, R) is a sufficient pair if and only if there is a finite $\kappa \geq 0$ so that (Q, R) is a $P_*(\kappa)$ pair. By extension, a P_* HLCP will be called a sufficient HLCP and a P_* pair will be called a sufficient pair.

Let us note that if (Q, R) is a sufficient pair, then the matrix $[Q\ R]$ is full rank. In fact, we have the following slightly stronger result.

Theorem 1 ([8]) *If Q and R are two $n \times n$ matrices such that the pair (Q, R) is column sufficient, then the matrix $[Q\ R]$ is full rank.*

The set of feasible points of the HLCP is denoted by

$$\mathcal{F} := \{z = \lceil x, s \rfloor \in R^{2n}_+ : Qx + Rs = b\}, \tag{6}$$

and the set of strictly feasible (or interior) points by

$$\mathcal{F}^0 := \{z = \lceil x, s \rfloor \in R^{2n}_{++} : Qx + Rs = b\}. \tag{7}$$

If the starting point is infeasible, then we will also consider the sets

$$\mathcal{F}_{\bar{b}} := \{\lceil z, \tau \rfloor = \lceil x, s, \tau \rfloor \in R^{2n+1}_{++} : Qx + Rs = b - \tau \bar{b}\}, \tag{8}$$

for appropriately chosen vectors $\bar{b}$. Let us note that for $\bar{b} = 0$, $\lceil z, \tau \rfloor \in \mathcal{F}_0$ implies $z \in \mathcal{F}^0$.

The solution set of HLCP is denoted by

$$\mathcal{F}^* := \{z = \lceil x, s \rfloor \in \mathcal{F} : xs = 0\}. \tag{9}$$

A solution $\lceil x^*, s^* \rfloor \in \mathcal{F}^*$ of HLCP is called *strictly complementary* if $x^* + s^* > 0$. The set of all strictly complementary solutions is denoted by

$$\mathcal{F}^c := \{z^* = \lceil x^*, s^* \rfloor \in \mathcal{F}^* : x^* + s^* > 0\}.$$

Not every HLCP has a strictly complementary solution. HLCP is called *nondegenerate* if it has a strictly complementary solution. Otherwise, it is called *degenerate*. Since better superlinear convergence results can be obtained when a problem is nondegenerate, we consider these two cases separately. However, we present a unified treatment of the two cases by introducing the parameter

$$\sigma := \begin{cases} 1 & \text{for general HLCP (default option),} \\ 0 & \text{if the HLCP is known to be nondegenerate.} \end{cases} \tag{10}$$

The following result of Stoer and Wechs [17] is of crucial importance in the design and analysis of our algorithm.

Theorem 2 ([17]) *Let HLCP be sufficient and let σ be defined as above. Assume that $\mathcal{F}^* \neq \emptyset$ and that there exists z such that $\lceil z, \tilde{\tau} \rfloor \in \mathcal{F}_{\bar{b}}$ for some $\tilde{\tau} > 0$ and $\bar{b}$ in $\mathbb{R}^n$. Then the system*

$$\begin{aligned} x(t,p)s(t,p) &= tp, \\ Qx(t,p) + Rs(t,p) &= b - t\bar{b} \end{aligned} \tag{11}$$

has a unique positive solution $z(t,p) = \lceil x(t,p), s(t,p) \rfloor$ for any $t \in (0, \tilde{\tau}]$ and any $p \in \mathbb{R}^n_{++}$. Moreover, the function $\tilde{z}(\rho, p) := z(\rho^{1+\sigma}, p)$ is an analytic function in $\rho = t^{1/(1+\sigma)}$ and p, that can be extended analytically to an open neighborhood of $[0, \tilde{\rho}] \times \mathbb{R}^n_{++}$, where $\tilde{\rho} = \tilde{\tau}^{1/(1+\sigma)}$. For any compact set $\mathcal{K} \subset \mathbb{R}^n_{++}$ and any integer $i \in \mathbb{N}$ there are constants $\bar{c}(\mathcal{K}, i)$ such that

$$\left\| \frac{\partial^i \tilde{z}(\rho, p)}{\partial \rho^i} \right\|_2 \le \bar{c}(\mathcal{K}, i), \quad \forall \rho \in [0, \tilde{\rho}], \ \forall p \in \mathcal{K}, i = 0, 1, 2, \ldots.$$

The curve $z = z(t, e)$, which is obtained by taking $p = e$ in (11), was called in [3] *the infeasible central path pinned on* $\bar{b}$. Such a path can be associated with any infeasible starting point $z^0 = \lceil x^0, s^0 \rfloor \in \mathbb{R}^{2n}_{++}$, by taking

$$\bar{b} := -\frac{r^0}{\tau_0}, \qquad r^0 := Qx^0 + Rs^0 - b, \qquad \tilde{\tau} := \tau_0 := \mu_0 := \mu(z^0) = \frac{x^{0T} s^0}{n}. \quad (12)$$

If the starting point is feasible (i.e., if $\bar{b} = 0$), then the curve $z = z(t, e)$ is the classical central path considered in the interior point literature.

3 A first order corrector-predictor method for sufficient HLCP

We first present Miao's generalization of MTY [10]. It was initially formulated for standard $P_*(\kappa)$ LCP, but in what follows we present it in the $P_*(\kappa)$ HLCP framework. At a typical iteration, one has a strictly feasible point z in the $\mathcal{N}_2(\alpha)$ neighborhood of the central path,

$$\mathcal{N}_2(\alpha) := \{z = \lceil x, s \rfloor \in \mathbb{R}^{2n}_{++} : \|xs - \mu(z)e\|_2 \leq \alpha \mu(z)\}. \quad (13)$$

The predictor step of the algorithm consists in computing a point $\bar{z}$ that belongs to a larger neighborhood $\mathcal{N}_2(\beta)$ of the central path, where

$$\beta = \frac{1 + 2(3\kappa + 1)\alpha}{3(2\kappa + 1)}. \quad (14)$$

The "predicted point" $\bar{z} = \lceil \bar{x}, \bar{s} \rfloor$ is obtained by first computing the affine scaling direction $w = \lceil u, v \rfloor$ as a solution of the linear system

$$\begin{cases} su + xv = -xs, \\ Qu + Rv = 0, \end{cases} \quad (15)$$

and by setting

$$\bar{z} = z + \bar{\theta} w, \quad \bar{\theta} := \max\{\hat{\theta} \in [0, 1] : z + \theta w \in \mathcal{N}_2(\beta), \forall \theta \in [0, \hat{\theta}]\}. \quad (16)$$

In the corrector step one computes the centering direction $\bar{w} = \lceil \bar{u}, \bar{v} \rfloor$,

$$\begin{cases} \bar{s}\bar{u} + \bar{x}\bar{v} = \bar{\mu} e - \bar{x}\bar{s}, \\ Q\bar{u} + R\bar{v} = 0, \end{cases} \quad (17)$$

where $\bar{\mu} = \mu(\bar{z}) = \bar{x}^T \bar{s}/n$, and defines the "corrected point" as

$$z^+ = z + \theta^+ \bar{w}, \quad \theta^+ = \frac{1 - \beta}{(3\kappa + 1)\beta}. \quad (18)$$

Since it can be proved that $z^+ \in \mathcal{N}_2(\alpha) \cap \mathcal{F}^0$, the predictor and corrector steps can be iterated leading to the following algorithm:

Algorithm 1 (First Order Predictor-Corrector)
Given real parameters $0 < \alpha < 1$, $\kappa \geq 0$, and a vector $z^0 \in \mathcal{N}_2(\alpha) \cap \mathcal{F}^0$:
Compute β from (14);
Set $k \leftarrow 0$;
repeat
Set $z \leftarrow z^k$;
(*predictor step*)
Compute $\bar{z}$ from (16);
(*corrector step*)
Compute z^+ from (18);
Set $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow \mu(z^+)$, $k \leftarrow k+1$.
continue

The results from [10] can be summarized in the following theorem:

Theorem 3 *If HLCP is $P_*(\kappa)$, then Algorithm 1 generates a sequence (z^k) of points belonging to $\mathcal{N}_2(\alpha)$ satisfying*

$$\mu_{k+1} \leq \left(1 - \frac{\sqrt{1-\alpha}}{2(1+3\kappa)\sqrt{3n}}\right)\mu_k, \quad k = 0, 1, \ldots. \tag{19}$$

Moreover, if HLCP is nondegenerate then the sequence (μ_k) converges Q-quadratically to zero.

For monotone HLCPs, i.e., when $\kappa = 0$, by taking $\alpha = 1/4$, we get $\beta = 1/2$ and $\theta^+ = 1$, so that Algorithm 1 reduces to the generalization of MTY for monotone LCPs from [6]. We note that the original MTY algorithm for LP from [11] also uses $\alpha = 1/4$, $\beta = 1/2$ and $\theta^+ = 1$. The HLCP corresponding to a LP is always nondegenerate and skew symmetric, in the sense that

$$Qu + Rv = 0 \text{ implies } u^T v = 0, \quad \text{for any } u, v \in \mathbb{R}^n. \tag{20}$$

In the skew-symmetric case, one can obtain finer estimates. For example, in [11] it is shown that

$$\mu_{k+1} \leq (1 - 1/(\sqrt[4]{8}\sqrt{n}))\mu_k, \quad k = 0, 1, \ldots \tag{21}$$

which is better than the estimate $\mu_{k+1} \leq (1 - 1/(4\sqrt{n}))\mu_k$ obtained by setting $\kappa = 0$ and $\alpha = 1/4$ in Theorem 3.

As evidenced by (14), (16), and (18), Algorithm 1 depends on the handicap κ of the HLCP, so that Algorithm 1 can be applied only to sufficient HLCPs for which the handicap, or at least an upper bound for the handicap, is known. In what follows, we will show that by interchanging the roles of the predictor and the corrector, we obtain an interior point method for sufficient HLCPs that does not depend on the handicap κ. The resulting algorithm is called a corrector-predictor algorithm.

In order to make the algorithm independent of κ we will produce at each iteration a triple $\lceil x, s, \tau \rfloor$ belonging to the following neighborhood of the central path:

$$\bar{\mathcal{N}}_2(\beta) := \{\lceil z, \tau \rfloor = \lceil x, s, \tau \rfloor \in \mathbb{R}^{2n+1}_{++} : \delta_2(z, \tau) \leq \beta\} \tag{22}$$

where

$$\delta_2(z,\tau) := \left\| \frac{xs}{\tau} - e \right\|_2. \tag{23}$$

The relation between the neighborhoods defined by (13) and (22) is obvious: $z \in \mathcal{N}_2(\alpha)$ if and only if $\lceil z, \mu(z) \rfloor \in \bar{\mathcal{N}}_2(\alpha)$. Considering the neighborhood $\bar{\mathcal{N}}_2(\beta)$ is very convenient since it generalizes easily to the case when the starting point is not feasible, as will be seen in the next sections. If the starting point is feasible and the HLCP is skew-symmetric, then our algorithm will produce points $\lceil z, \tau \rfloor \in \bar{\mathcal{N}}_2(\beta)$ with $\tau = \mu(z)$, so that in this case all the points z generated by the algorithm will belong to $\mathcal{N}_2(\beta)$. In general $\tau \neq \mu(z)$ but, as shown by the following lemma, τ and $\mu(z)$ are very close.

Lemma 1 *If* $\lceil z, \tau \rfloor = \lceil x, s, \tau \rfloor \in \bar{\mathcal{N}}_2(\beta)$, *then the following inequalities are satisfied*

$$(1-\beta)\tau \le x_i s_i \le (1+\beta)\tau, \quad \forall i = 1, \ldots, n, \tag{24}$$

$$\left(1 - \frac{\beta}{\sqrt{n}}\right)\tau \le \mu \le \left(1 + \frac{\beta}{\sqrt{n}}\right)\tau. \tag{25}$$

Proof The first inequality follows from the fact that $|x_i s_i - \tau| \le \|xs - \tau e\|_2 \le \beta\tau$, while the second inequality follows from

$$\beta^2\tau^2 \ge \|xs - \tau e\|_2^2 = \|xs - \mu e\|_2^2 + n(\tau - \mu)^2 \ge n(\tau - \mu)^2.$$

In deducing the above relation, we have used the fact that the vector μe is the orthogonal projection of the vector xs on $Span(e)$.

At a typical iteration of our algorithm, we have a point $\lceil z, \tau \rfloor \in \bar{\mathcal{N}}_2(\beta)$, with $z \in \mathcal{F}^0$, where β can be any real number such that $0 < \beta < 1$. In contrast with Algorithm 1, we start with a corrector step in order to improve centrality. This is done by computing the centering direction $w = \lceil u, v \rfloor$ as the solution of the linear system

$$\begin{cases} su + xv = \tau e - xs, \\ Qu + Rv = 0, \end{cases} \tag{26}$$

and defining

$$z(\theta) = \lceil x(\theta), s(\theta) \rfloor := z + \theta w, \tag{27}$$

$$\bar{\theta} = \operatorname{argmin} \delta_2(z(\theta), \tau), \tag{28}$$

$$\bar{z} := \lceil \bar{x}, \bar{s} \rfloor := z + \bar{\theta} w, \bar{\tau} = \tau. \tag{29}$$

From construction, the corrector produces a point $\lceil \bar{z}, \bar{\tau} \rfloor \in \bar{\mathcal{N}}_2(\bar{\beta})$, with $\bar{\beta} < \beta$.

In the predictor step we improve optimality by computing the affine scaling direction $\bar{w} = \lceil \bar{u}, \bar{v} \rfloor$ from

$$\begin{cases} \bar{s}\bar{u} + \bar{x}\bar{v} = -\bar{x}\bar{s}, \\ Q\bar{u} + R\bar{v} = 0, \end{cases} \tag{30}$$

and setting

$$\bar{z}(\theta) = \lceil \bar{x}(\theta), \bar{s}(\theta) \rfloor := \bar{z} + \theta \bar{w}, \bar{\tau}(\theta) = (1-\theta)\bar{\tau}, \tag{31}$$

$$\theta^+ := \max\{\hat{\theta} \in [0,1] : \lceil \bar{z}(\theta), \bar{\tau}(\theta) \rfloor \in \bar{\mathcal{N}}_2(\beta), \forall \theta \in [0, \hat{\theta}]\}, \tag{32}$$

$$z^+ := \bar{z}(\theta^+), \tau_+ := \tau(\theta^+). \tag{33}$$

Since $\lceil z^+, \tau_+ \rfloor \in \bar{\mathcal{N}}_2(\beta)$ and $z^+ \in \mathcal{F}^0$, we can iterate the above scheme and obtain the following algorithm.

Algorithm 2 (First Order Corrector-Predictor)
Given a real parameter $0 < \beta < 1$ and a vector $z^0 \in \mathcal{N}_2(\beta) \cap \mathcal{F}^0$:
  Set $\tau_0 \leftarrow \mu(z^0)$;
  Set $k \leftarrow 0$;
  repeat
    Set $\lceil z, \tau \rfloor \leftarrow \lceil z^k, \tau_k \rfloor$;
      (*corrector step*)
      Compute $\lceil \bar{z}, \bar{\tau} \rfloor$ from (26)–(29);
      (*predictor step*)
      Compute $\lceil z^+, \tau_+ \rfloor$ from (30)–(33);
    Set $\lceil z^{k+1}, \tau_{k+1} \rfloor \leftarrow \lceil z^+, \tau_+ \rfloor, k \leftarrow k+1$.
  continue

In what follows, we will show that the above algorithm, which is independent of κ, has the same desirable properties as Algorithm 1.

First, we find bounds for the solution of a linear system of the form

$$\begin{aligned} su + xv &= a, \\ Qu + Rv &= 0. \end{aligned} \tag{34}$$

Using the notation

$$D = X^{-1/2}S^{1/2}, \qquad \|\lceil u, v \rfloor\|_z^2 := \|Du\|_2^2 + \|D^{-1}v\|_2^2, \qquad \tilde{a} = (xs)^{-1/2}a, \tag{35}$$

we obtain the following result. □

Lemma 2 *If HLCP is $P_*(\kappa)$, then for any $z = \lceil x, s \rfloor \in \mathbb{R}^{2n}_{++}$ and any $a \in \mathbb{R}^n$ the linear system (34) has a unique solution $w = \lceil u, v \rfloor$ for which the following estimates hold*

$$\|w\|_z^2 \leq (1+2\kappa)\|\tilde{a}\|_2^2, \|uv\|_2 \leq \left(\frac{1}{\sqrt{8}} + \kappa\right)\|\tilde{a}\|_2^2, -\kappa\|\tilde{a}\|_2^2 \leq u^T v \leq \frac{1}{4}\|\tilde{a}\|_2^2.$$

Proof We consider the index sets:

$$\mathcal{I}^+ = \{i : u_i v_i > 0\}, \ \mathcal{I}^- = \{i : u_i v_i < 0\}.$$

Using the relations

$$0 < 4u_i v_i \le (Du + D^{-1}v)_i^2 = \tilde{a}_i^2,$$

$$\sum_{i\in\mathcal{I}^-} |u_i v_i| \le (1+4\kappa) \sum_{i\in\mathcal{I}^+} u_i v_i \le \left(\frac{1}{4}+\kappa\right)\|\tilde{a}\|_2^2$$

and applying Lemmas 3.2 and 3.3 from [13] we deduce that

$$-\kappa\|\tilde{a}\|_2^2 \le u^T v = \sum_{i\in\mathcal{I}^-} u_i v_i + \sum_{i\in\mathcal{I}^+} u_i v_i \le \sum_{i\in\mathcal{I}^+} u_i v_i \le \frac{1}{4}\|\tilde{a}\|_2^2. \tag{36}$$

From (34), (35) and (36) it follows that

$$\begin{aligned}\|w\|_z^2 &= \|Du\|_2^2 + \|D^{-1}v\|_2^2 = \|Du + D^{-1}v\|_2^2 - 2u^T v\\ &= \|\tilde{a}\|_2^2 - 2u^T v \le (1+2\kappa)\|\tilde{a}\|_2^2\end{aligned}$$

and

$$\begin{aligned}\|uv\|_2^2 &= \sum_{i\in\mathcal{I}^+} u_i^2 v_i^2 + \sum_{i\in\mathcal{I}^-} u_i^2 v_i^2 \le \frac{1}{16}\sum_{i\in\mathcal{I}^+} \tilde{a}_i^4 + \left(\sum_{i\in\mathcal{I}^-} u_i v_i\right)^2\\ &\le \frac{1}{16}\|\tilde{a}\|_2^4 + \left(\frac{1}{4}+\kappa\right)^2\|\tilde{a}\|_2^4 = \left(\frac{1}{8}+\frac{\kappa}{2}+\kappa^2\right)\|\tilde{a}\|_2^4\\ &\le \left(\frac{1}{\sqrt{8}}+\kappa\right)^2\|\tilde{a}\|_2^4.\end{aligned}$$

□

In the next theorem we will prove the convergence of Algorithm 2. In the statement and the proof of this theorem we will use the following notation:

$$\varpi := \begin{cases} 1+\beta/\sqrt{n} & \text{for general HLCP,}\\ 1 & \text{if the HLCP is skew-symmetric,}\end{cases} \tag{37}$$

$$\bar{\psi}(\alpha) = \frac{\sqrt{8}}{1+\sqrt{1+8\alpha}}, \tag{38}$$

$$\alpha_0(\beta) = \frac{\sqrt{2}+\beta}{\sqrt{2}(1-\beta)\beta(\sqrt{8}-\beta)}, \tag{39}$$

$$\alpha_1(\beta) = \frac{\sqrt{2}(1-\beta)}{\beta(4-(4+\sqrt{2})\beta)}. \tag{40}$$

Theorem 4 *If HLCP is $P_*(\kappa)$, then Algorithm 2 generates a sequence of points $(\lceil z^k, \tau_k\rfloor)$ belonging to $\bar{\mathcal{N}}_2(\beta)$ and there is a constant $c = c(\beta) > 0$, depending only on β, such that*

$$\tau_{k+1} \le \left(1 - \frac{c(\beta)}{(1+\sqrt{8}\kappa)\sqrt{n}}\right)\tau_k, \quad k = 0, 1, \ldots, \tag{41}$$

$$c(\beta) \geq \bar{\psi}(\alpha_0(\beta)), \quad \forall \beta \in (0,1). \tag{42}$$

Moreover, if HLCP is nondegenerate then the sequences $(\mu(z^k))$ *and* (τ_k) *converge Q-quadratically to zero.*

If HLCP is skew symmetric, then the following finer estimate holds

$$c(\beta) \geq \bar{\psi}(\alpha_1(\beta)), \quad \forall \beta \in (0, 2/(2+\sqrt{2})). \tag{43}$$

Proof First, we quantify the centering effect of the corrector step. We note that from (26)–(29) it follows that

$$x(\theta)s(\theta) = (1-\theta)xs + \theta\tau e + \theta^2 uv. \tag{44}$$

Since $\lceil z, \tau \rfloor \in \bar{\mathcal{N}}_2(\beta)$, using a continuity argument it is easy to show that $\bar{z} \in \mathcal{F}^0$. If $0 \leq \theta \leq 1$, then using (44) we deduce that

$$\|x(\theta)s(\theta) - \tau e\|_2 \leq (1-\theta)\|xs - \tau e\|_2 + \theta^2\|uv\|_2 \leq (1-\theta)\beta\tau + \theta^2\|uv\|_2,$$

We apply Lemma 2 to the case $a = \tau e - xs$. Since $\lceil z, \tau \rfloor \in \bar{\mathcal{N}}_2(\beta)$, we have

$$\|\tilde{a}\|_2^2 \leq \frac{\beta^2\tau}{1-\beta}, \qquad \|uv\|_2 \leq \frac{(1+\sqrt{8}\kappa)\beta^2\tau}{\sqrt{8}(1-\beta)}, \qquad -\frac{\kappa\beta^2\tau}{1-\beta} \leq u^T v \leq \frac{.25\beta^2\tau}{1-\beta}$$

so that

$$\delta_2(z(\theta), \tau) \leq g_1(\theta) := (1-\theta)\beta + \frac{(1+\sqrt{8}\kappa)\beta^2}{\sqrt{8}(1-\beta)}\theta^2.$$

It is easily seen that

$$\widetilde{\theta}_1 := \operatorname*{argmin}_{\theta\in[0,1]} g_1(\theta) = \min\left\{1, \frac{\sqrt{2}(1-\beta)}{(1+\sqrt{8}\kappa)\beta}\right\}.$$

Since g_1 is decreasing on $[0, \widetilde{\theta}_1]$ and

$$\widehat{\theta}_1 := \frac{1-\beta}{1+\sqrt{8}\kappa} < \widetilde{\theta}_1,$$

we have

$$\beta - \bar{\beta} := \beta - \delta_2(z(\bar{\theta}), \tau) \geq \beta - g_1(\widetilde{\theta}_1) > \beta - g_1(\widehat{\theta}_1) = \frac{(1-\beta)\beta(1-\frac{\beta}{\sqrt{8}})}{1+\sqrt{8}\kappa}. \tag{45}$$

If $\kappa = 0$ and $\beta \leq 2/(2+\sqrt{2})$ then $\widetilde{\theta}_1 = 1$, so that

$$\beta - \bar{\beta} \geq \beta - g_1(1) = \frac{\beta(1-\beta(1+\frac{1}{\sqrt{8}}))}{1-\beta}. \tag{46}$$

Let us now quantify the improvement in optimality achieved by the predictor step (30)–(33). We first note that from (30) and (31) it follows that

$$\bar{x}(\theta)\bar{s}(\theta) = (1-\theta)\bar{x}\bar{s} + \theta^2\bar{u}\bar{v}, \qquad \bar{\tau}(\theta) = (1-\theta)\bar{\tau} = (1-\theta)\tau. \tag{47}$$

Therefore for any $\theta \in [0, 1]$ we have

$$\|\bar{x}(\theta)\bar{s}(\theta) - \bar{\tau}(\theta)e\|_2 \leq (1-\theta)\|\bar{x}\bar{s} - \bar{\tau}e\|_2 + \theta^2\|\bar{u}\bar{v}\|_2 \leq (1-\theta)\bar{\beta}\bar{\tau} + \theta^2\|\bar{u}\bar{v}\|_2.$$

By applying Lemma 2 to the case $z = \bar{z}$, $a = -\bar{x}\bar{s}$, and Lemma 1 for $\lceil\bar{z}, \bar{\tau}\rfloor \in \bar{\mathcal{N}}_2(\bar{\beta}) \subset \bar{\mathcal{N}}_2(\beta)$, we get

$$\|\tilde{a}\|_2^2 = n\bar{\mu} \leq n\varpi\bar{\tau}, \qquad \|\bar{u}\bar{v}\|_2 \leq (1/\sqrt{8} + \kappa)n\bar{\mu} \leq (1/\sqrt{8} + \kappa)n\varpi\bar{\tau}.$$

Therefore the following implication holds

$$(1-\theta)\bar{\beta} + \theta^2(1/\sqrt{8} + \kappa)\varpi n \leq \beta(1-\theta) \quad \Rightarrow \quad \lceil\bar{z}(\theta), \bar{\tau}(\theta)\rfloor \in \bar{\mathcal{N}}_2(\beta).$$

The inequality on the left-hand-side can be written as $g(\theta) := 1 - \theta - \bar{\alpha}\theta^2 \geq 0$, where,

$$\bar{\alpha} = \frac{\varpi(1+\sqrt{8}\kappa)n}{\sqrt{8}(\beta - \bar{\beta})} \leq \frac{(\sqrt{2}+\beta)(1+\sqrt{8}\kappa)n}{4(\beta - \bar{\beta})}, \quad \forall n \geq 2. \tag{48}$$

It is easily seen that

$$g(\theta) \geq 0, \quad \forall 0 \leq \theta \leq \bar{\theta}(\bar{\alpha}) := \frac{2}{1 + \sqrt{1 + 4\bar{\alpha}}}.$$

From the definition (32) of θ^+ it follows that $\theta^+ \geq \bar{\theta}(\bar{\alpha})$. It is easily seen that $\bar{\theta}(\alpha)$ is decreasing in α and that for any $\nu \geq 2$ there holds

$$\bar{\theta}(\nu\alpha) \geq \frac{\bar{\psi}(\alpha)}{\sqrt{\nu}}, \tag{49}$$

where the function $\bar{\psi}$ is defined in (38).

From (45), (48) and (39) it follows that for any $n \geq 2$, $\kappa \geq 0$ and $\beta \in (0, 1)$ we have

$$\bar{\alpha} \leq \frac{\varpi(1+\sqrt{8}\kappa)^2 n}{(1-\beta)\beta(\sqrt{8}-\beta)} \leq \frac{(\sqrt{2}+\beta)(1+\sqrt{8}\kappa)^2 n}{\sqrt{2}(1-\beta)\beta(\sqrt{8}-\beta)} = \alpha_0(\beta)(1+\sqrt{8}\kappa)^2 n.$$

Finally, if HLCP is skew symmetric and $0 < \beta < 2/(2+\sqrt{2})$, then by taking $\kappa = 0$ and $\varpi = 1$ in (48) and using (46) we get

$$\bar{\alpha} \leq \frac{\sqrt{2}(1-\beta)n}{\beta(4 - (4+\sqrt{2})\beta)} = \alpha_1(\beta)n,$$

where $\alpha_1(\beta)$ is defined by (40). Inequalities (42) and (43) follow then from (49). The quadratic convergence is easily proved using Theorem 2. However we will not give the proof here, since the result is a particular case of a more general result to be proved in Sect. 4.3.

We note that if HLCP is skew symmetric then the sequences (μ_k) and (τ_k), where $\mu_k = \mu(z^k)$, coincide so that in this case we have

$$\mu_{k+1} \leq \left(1 - \frac{\bar{\psi}(\alpha_1(\beta))}{\sqrt{n}}\right)\mu_k, \quad k = 0, 1, \ldots, \ \forall \beta \in (0, 2/(2+\sqrt{2})).$$

The above estimates are to be compared with the estimates (21) obtained in [11] for the original MTY predictor-corrector method for LP. Since

$$\bar{\psi}(\alpha_1(1/2)) = \frac{2\sqrt{2}}{1 + \sqrt{\frac{4+15\sqrt{2}}{4-\sqrt{2}}}} = 0.686077\ldots > \frac{1}{\sqrt[4]{8}} = 0.594604\ldots$$

it follows that the estimates that are obtained from Theorem 4 are better than the estimates (21). As mentioned in the first part of this section, the estimates (21) are better than the estimates that can be obtained from Theorem 3. However, in general it is not so easy to compare the estimates given in Theorems 3 and 4 since (19) deals with the sequence (μ_k) while (41) with the sequence (τ_k). As seen below, both theorems imply $O((1+\kappa)\sqrt{n}L)$ iteration complexity, but the constant involved in the $O(.)$ notation is smaller for Algorithm 2. □

Corollary 1 *If HLCP is sufficient then for any $\varepsilon > 0$ the sequence (z^k) produced by Algorithm 2 satisfies*

$$x^{kT} s^k \leq \varepsilon, \quad \forall k \geq K_\varepsilon,$$

with

$$K_\varepsilon = \left\lceil \frac{1 + \frac{\beta}{\sqrt{n}L_\varepsilon}}{c(\beta)}(1 + \sqrt{8}\kappa)\sqrt{n}L_\varepsilon \right\rceil, \quad L_\varepsilon = \log \frac{x^{0T} s^0}{\varepsilon}.$$

Proof From (25) and (41) we have

$$x^{kT} s^k = n\mu_k \leq n\left(1 + \frac{\beta}{\sqrt{n}}\right)\tau_k \leq n\left(1 + \frac{\beta}{\sqrt{n}}\right)\left(1 - \frac{c(\beta)}{(1+\sqrt{8}\kappa)\sqrt{n}}\right)^k \tau_0$$
$$= x^{0T} s^0 \left(1 + \frac{\beta}{\sqrt{n}}\right)\left(1 - \frac{c(\beta)}{(1+\sqrt{8}\kappa)\sqrt{n}}\right)^k.$$

It follows that $x^{k^T} s^k \leq \varepsilon$ whenever

$$k \log\left(1 - \frac{c(\beta)}{(1+\sqrt{8}\kappa)\sqrt{n}}\right) \leq \log \frac{\varepsilon}{x^{0T} s^0 (1 + \frac{\beta}{\sqrt{n}})}.$$

Since $\log(1-t) \le -t$, the above inequality is satisfied if

$$k \ge \frac{(1+\sqrt{8}\kappa)\sqrt{n}}{c(\beta)} \log \frac{x^{0T}s^0(1+\frac{\beta}{\sqrt{n}})}{\varepsilon},$$

and the desired result follows by noticing that

$$\frac{x^{0T}s^0(1+\frac{\beta}{\sqrt{n}})}{\varepsilon} = \exp(L_\varepsilon)\left(1+\frac{\beta}{\sqrt{n}}\right) \le \exp\left(L_\varepsilon + \frac{\beta}{\sqrt{n}}\right). \qquad \square$$

Using Theorem 3 we can prove in a similar manner that for the sequence (z^k) generated by Algorithm 1 we have

$$x^{k^T}s^k \le \varepsilon, \quad \forall k \ge \bar{K}_\varepsilon := \left\lceil \frac{2\sqrt{3}}{\sqrt{1-\alpha}}(1+3\kappa)\sqrt{n}L_\varepsilon \right\rceil.$$

By comparing this result with the result obtained in Corollary 1 we first remark that the coefficient of κ in K_ε is slightly smaller than the corresponding coefficient in $\bar{K}_\varepsilon$, ($\sqrt{8}$ instead of 3). We also remark that

$$\frac{2\sqrt{3}}{\sqrt{1-1/4}} = 4, \qquad \frac{2\sqrt{3}}{\sqrt{1-\alpha}} > 2\sqrt{3} = 3.4641\ldots, \quad \forall \alpha \in (0,1),$$

while, if $\sqrt{n}L_\varepsilon \ge \bar{\nu}$

$$\frac{1+\frac{\beta}{\sqrt{n}L_\varepsilon}}{c(\beta)} \le \frac{1+\beta/\bar{\nu}}{\bar{\psi}(\alpha_0(\beta))} =: c_0(\beta,\bar{\nu}), \quad \forall \beta \in (0,1),\ \forall \bar{\nu} \le \sqrt{n}L_\varepsilon.$$

The factor $c_0(\beta,\bar{\nu})$ is decreasing in $\bar{\nu}$ and we are interested in computational complexity results for large values of $\sqrt{n}L_\varepsilon$. However, even for small values of $\bar{\nu}$, we can improve the coefficient 3.4641 above. Thus we have

$$c_0(.5,1) < 2.88, \qquad c_0(.5,2) < 2.40, \qquad c_0(.5,6) < 2.01, \qquad c_0(.5,12) < 2.00.$$

In conclusion, we have developed a corrector-predictor method for sufficient HCLP that does not depend on the handicap κ of the problem, and in the same time we have obtained computational complexity estimates that improve the existing estimates for predictor-corrector methods. In the next section we will generalize this corrector-predictor method to the case where a strictly feasible starting point is not available. We will also consider higher order variants of the corrector-predictor method that are superlinearly convergent even if the HLCP is degenerate.

4 A higher order infeasible corrector-predictor algorithm

The infeasible interior point method to be analyzed in this section follows approximately the *infeasible central path pinned on* $\bar{b}$, where $\bar{b}$ is defined by (12). We identify this path with the curve

$$\mathcal{C}_{\bar{b}} = \{\lceil x, s, \tau \rfloor \in \mathcal{F}_{\bar{b}} : xs = \tau p\}, \tag{50}$$

where $\mathcal{F}_{\bar{b}}$ is defined in (8). We observe that for any point $\lceil z, \tau \rfloor$ belonging to the central path $\mathcal{C}_{\bar{b}}$, we have $\mu(z) = \tau$ and $r(z) = Qx + Rs - b = -\tau \bar{b}$. The iterates produced by the infeasible interior-point method proposed in this paper belong to the neighborhood $\bar{\mathcal{N}}_2(\beta)$ of $\mathcal{C}_{\bar{b}}$. If $\lceil z, \tau \rfloor \in \bar{\mathcal{N}}_2(\beta) \cap \mathcal{F}_{\bar{b}}$ then $r(z) = -\tau \bar{b}$, but in general $\mu(z) \neq \tau$. However μ (which is a measure of optimality) and τ (which is measure of feasibility) are very close as indicated by (25).

At the start of a typical iteration, we have a point $\lceil z, \tau \rfloor \in \bar{\mathcal{N}}_2(\beta) \cap \mathcal{F}_{\bar{b}}$.

First, we perform a corrector step in order to improve the centrality of $\lceil z, \tau \rfloor$. We use the same corrector step (26)–(29) as in Algorithm 2 and obtain a point $\lceil \bar{z}, \bar{\tau} \rfloor \in \bar{\mathcal{N}}_2(\bar{\beta})$, with $\bar{\tau} = \tau$ and $\bar{\beta} < \beta$.

Second, we perform the predictor step to increase optimality and feasibility by moving along a high-order curve

$$\bar{z}(\theta) = \bar{z} + \theta w^1 + \cdots + \theta^m w^m, \tag{51}$$

where the vectors $w^i = \lceil u^i, v^i \rfloor$ are obtained by solving the following m linear systems

$$\begin{aligned}
&\begin{cases} \bar{s}u^1 + \bar{x}v^1 = -(1+\sigma)\bar{x}\bar{s}, \\ Qu^1 + Rv^1 = (1+\sigma)\tau\bar{b}, \end{cases} \\
&\begin{cases} \bar{s}u^2 + \bar{x}v^2 = \sigma\bar{x}\bar{s} - u^1 v^1, \\ Qu^2 + Rv^2 = -\sigma\tau\bar{b}, \end{cases} \\
&\quad \cdots \\
&\begin{cases} \bar{s}u^i + \bar{x}v^i = -\sum_{j=1}^{i-1} u^j v^{i-j}, \\ Qu^i + Rv^i = 0, \end{cases} \qquad i = 3, 4, \ldots, m.
\end{aligned} \tag{52}$$

It can be shown that $\bar{z}(\theta)$ coincides with the m'th order Taylor expansion of the mapping $\theta \to \tilde{z}((1-\theta)\tau^{1/(1+\sigma)}, p)$, where $\tilde{z}(\rho, p)$ is the function defined in Theorem 2 with $p = \bar{x}\bar{s}/\tau$ and $\bar{b}$ given by (12). Indeed, from m consecutive differentiations of the nonlinear system

$$\tilde{x}(\rho, p)\tilde{s}(\rho, p) = \rho^{1+\sigma} p,$$

$$Q\tilde{x}(\rho, p) + R\tilde{s}(\rho, p) = b - \rho^{1+\sigma}\bar{b}$$

it follows that

$$w^i = \lceil u^i, v^i \rfloor = \frac{(-1)^i}{i!}\tau^{i/(1+\sigma)} \frac{\partial^i}{\partial \rho^i} \tilde{z}(\rho, (\bar{x}\bar{s})/\tau)|_{\rho = \tau^{1/(1+\sigma)}}, \quad i = 1, \ldots, m.$$

The step length in the predictor step is given by

$$\theta^+ := \max\{\tilde{\theta} \in [0, 1] : \lceil \bar{z}(\theta), \bar{\tau}(\theta) \rfloor \in \bar{\mathcal{N}}_2(\beta), \forall \theta \in [0, \tilde{\theta}]\}, \tag{53}$$

where

$$\bar{\tau}(\theta) = (1-\theta)^{1+\sigma}\bar{\tau} = (1-\theta)^{1+\sigma}\tau. \tag{54}$$

As a result of the predictor step we obtain a point $\lceil z^+, \tau^+ \rfloor$ given by

$$z^+ = \bar{z}(\theta^+), \qquad \tau^+ = \bar{\tau}(\theta^+) = (1-\theta^+)^{1+\sigma}\tau. \tag{55}$$

From the definition of $\bar{z}(\theta)$ it follows that

$$\bar{x}(\theta)\bar{s}(\theta) = (1-\theta)^{1+\sigma}\bar{x}\bar{s} + \sum_{i=m+1}^{2m} \theta^i h^i, \quad \text{where } h^i := \sum_{j=i-m}^{m} u^j v^{i-j}. \tag{56}$$

We also have

$$\bar{x}(\theta)\bar{s}(\theta) - \bar{\tau}(\theta)e = (1-\theta)^{1+\sigma}(\bar{x}\bar{s} - \tau e) + \sum_{i=m+1}^{2m} \theta^i h^i, \tag{57}$$

$$\bar{r}(\theta) = (1-\theta)^{1+\sigma}\bar{r}. \tag{58}$$

Define $z^+ := \bar{z}(\theta^+)$, $\mu^+ := \bar{\mu}(\theta^+)$, $\tau^+ := (1-\theta^+)^{1+\sigma}\tau$ and $r^+ := Qx^+ + Rs^+ - b$. Note that $r^+ = (1-\theta^+)^{1+\sigma}\bar{r}$, that is, the residual is decreased in the predictor step by the same factor as τ. By construction, we have $\lceil z^+, \tau^+ \rfloor \in \bar{\mathcal{N}}_2(\beta)$, implying that the correction-prediction scheme can be repeated to obtain the following algorithm:

Algorithm 3 (m-th Order Corrector-Predictor)

Given a real number $0 < \beta < 1$ and a staring point $z^0 \in \mathcal{N}_2(\beta)$:
  Choose σ as indicated in (10);
  Choose an integer $m \geq 1$ such that $(\sigma, m) \neq (1, 1)$;
  Set, $\tau_0 \leftarrow \mu(z^0)$, $\mu_0 \leftarrow \mu(z^0)$;
  Set $k \leftarrow 0$;
  repeat
    Set $\lceil z, \tau \rfloor \leftarrow \lceil z^k, \tau_k \rfloor$;
      (*corrector step*)
      Compute $\lceil \bar{z}, \bar{\tau} \rfloor$ from (26)–(29);
      (*predictor step*)
      Compute $\lceil z^+, \tau^+ \rfloor$ from (51)–(55);
    Set $\lceil z^{k+1}, \tau_{k+1} \rfloor \leftarrow \lceil z^+, \tau^+ \rfloor$, $\mu_{k+1} \leftarrow \mu(z^+)$, $k \leftarrow k+1$.
  continue

The m-th order algorithm does not depend on the handicap κ. Even though each predictor step requires solving m linear systems, one needs to perform only one matrix factorization, since the system matrices in (52) are the same. In the next theorem, we show that the algorithm is well defined.

Theorem 5 *Let HLCP be sufficient and assume it has a solution. Then, Algorithm* 3 *is well-defined and the following properties are satisfies for all* $k = 0, 1, \ldots$

$$\begin{aligned} \lceil z^k, \tau_k \rfloor &\in \bar{\mathcal{N}}_2(\beta), \\ Qx^k + Rs^k &= b + \frac{\tau_k}{\tau_0} r^0, \\ \tau_{k+1} &= (1-\theta_k)^{1+\sigma} \tau_k, \\ r^{k+1} &= (1-\theta_k)^{1+\sigma} r^k. \end{aligned}$$

Proof All properties follow immediately from the definition of Algorithm 3. We only note that in the predictor step, we have $\delta_2(\bar{z}(\theta), \bar{\tau}(\theta)) \le \beta$ for all $\theta \in [0, \theta^+]$ and this implies $\bar{z}(\theta) > 0$, for all $\theta \in [0, \theta^+]$. Indeed if one of the components of $\bar{z}(\theta)$ (say $\bar{x}_i(\theta)$) vanishes, then

$$1 = \left| \frac{\bar{x}_i(\theta)\bar{s}_i(\theta)}{(1-\theta)\tau} - 1 \right| \le \delta_2(\bar{z}(\theta), \bar{\tau}(\theta)) \le \beta,$$

which contradicts the fact that $\beta < 1$. □

4.1 Technical results

We start this section by stating several basic results. Most of them have been known in one form or another in the interior point literature. We are using the formulations from [14].

Lemma 3 *If HLCP is* $P_*(\kappa)$, *then for any* $z = \lceil x, s \rfloor \in \mathbb{R}^{2n}_{++}$ *and any* $a, \tilde{b} \in \mathbb{R}^n$ *the linear system*

$$\begin{cases} su + xv = a, \\ Qu + Rv = \tilde{b} \end{cases}$$

has a unique solution $w = \lceil u, v \rfloor$ *and the following inequalities are satisfied*:

$$\|w\|_z \le \sqrt{1+2\kappa}\,\|\tilde{a}\|_2 + (1+\sqrt{2+4\kappa})\zeta(z, \tilde{b}), \qquad \|uv\|_2 \le \frac{1}{2}\|w\|_z^2,$$

where $\tilde{a}$ *is given by* (35), *and*

$$\zeta(z, \tilde{b})^2 = \min\{\|\lceil \tilde{u}, \tilde{v} \rfloor\|_z^2 : Q\tilde{u} + R\tilde{v} = \tilde{b}\} = \tilde{b}^T \left(QD^{-2}Q^T + RD^2R^T\right)^{-1} \tilde{b}.$$

Lemma 4 *If HLCP is sufficient then for any* $z = \lceil x, s \rfloor, z^0 = \lceil x^0, s^0 \rfloor \in \mathbb{R}^{2n}_{++}$ *we have*

$$\zeta(z, \tilde{b}) \le \left\| (xsx^0s^0)^{-1/2} \right\|_\infty (x^T s^0 + s^T x^0) \zeta(z^0, \tilde{b}),$$

where ζ *is defined in Lemma* 3.

Lemma 5 *Assume that HLCP is $P_*(\kappa)$ and that $\mathcal{F}^*$ is nonempty. Let $(x^0, s^0) \in \mathbb{R}^{2n}_{++}$ be a starting point and consider the notations from (8) and (12). Then for all $(x^*, s^*) \in \mathcal{F}^*$ and $\lceil z, \tau \rfloor = \lceil x, s, \tau \rfloor \in \mathcal{F}_{\bar{b}}$, with $0 < \tau < \tau_0$, we have*

$$x^T s^0 + s^T x^0 + \frac{\tau_0 - \tau}{\tau}(x^T s^* + s^T x^*) \leq (1 + 4\kappa)\big(\max\{1, \zeta^*\} + \mu(z)/\tau\big)n\tau_0,$$

where $\zeta^* = ((x^0)^T s^* + (s^0)^T x^*)/((x^0)^T s^0)$.

4.2 Global convergence and polynomial complexity

The results obtained in Lemma 5 will be used in the proof of the global convergence of Algorithm 3. While we will prove global convergence for general infeasible positive starting points, we will give polynomial complexity results only if the starting point is large enough in the sense that

$$z^0 \geq \gamma z^* \quad \text{for some } z^* \in \mathcal{F}^*, \tag{59}$$

where γ is a given positive constant, or almost feasible in the sense that the following infeasibility measure

$$\phi(z^0) = \min\{\|(z^0)^{-1}(z^0 - \widetilde{z})\|_\infty : (QR)\widetilde{z} = b\}, \tag{60}$$

is small enough, in the sense that

$$\phi(z^0) \leq \frac{2\sqrt{(1+2\kappa)(1-\beta)}}{(\max\{1, \zeta^*\} + 1 + \beta/\sqrt{2})(1 + \sqrt{2+4\kappa})(1+4\kappa)\sqrt{n}}, \tag{61}$$

for some $z^* \in \mathcal{F}^*$, where ζ^* is defined in Lemma 5. We note that the above inequality is trivially satisfied for feasible starting points, since for such points $\phi(z^0) = 0$. In order to treat the two cases together it is convenient to define

$$\chi = \begin{cases} 1, & \text{if (59) is satisfied,} \\ 0, & \text{if (61) is satisfied.} \end{cases} \tag{62}$$

In the next proposition we find an upper bound of the quantity $\zeta(z, \tau\bar{b})$, where $\bar{b}$ is defined in (12), in the terms of the infeasibility measure (60) of the starting point.

Proposition 1 *Assume that HLCP is $P_*(\kappa)$ and that $\mathcal{F}^*$ is nonempty. Let $(x^0, s^0) \in \mathbb{R}^{2n}_{++}$ be a starting point and consider the notations from (8) and (12). If $\lceil x^0, s^0, \tau_0 \rfloor \in \bar{\mathcal{N}}_2(\beta)$, then for all $z^* = \lceil x^*, s^* \rfloor \in \mathcal{F}^*$ and $\lceil z, \tau \rfloor = \lceil x, s, \tau \rfloor \in \mathcal{F}_{\bar{b}} \cap \bar{\mathcal{N}}_2(\beta)$, with $0 < \tau < \tau_0$, we have*

$$\zeta(z, \tau\bar{b}) \leq \bar{\zeta}\sqrt{\tau}, \quad \textit{where } \bar{\zeta} = \frac{\max\{1, \zeta^*\} + 1 + \beta/\sqrt{2}}{\sqrt{1-\beta}}\phi(z^0)(1+4\kappa)n\cdot$$

Proof For any $\widetilde{z} = \lceil \widetilde{x}, \widetilde{s} \rfloor$ satisfying $(QR)\widetilde{z} = b$ we have $(QR)(z^0 - \widetilde{z}) = r^0$, so that we can write

$$\zeta(z, \tau\bar{b}) = \frac{\tau}{\tau_0}\zeta(z, r^0) \le \frac{\tau}{\tau_0}\|z^0 - \widetilde{z}\|_z = \frac{\tau}{\tau_0}\left(\sum_{i=1}^{n} \frac{s_i}{x_i}(x_i^0 - \widetilde{x}_i)^2 + \frac{x_i}{s_i}(s_i^0 - \widetilde{s}_i)^2\right)^{1/2}$$
$$= \frac{\tau}{\tau_0}\left(\sum_{i=1}^{n} \frac{(s_i x_i^0)^2}{x_i s_i}\left(\frac{x_i^0 - \widetilde{x}_i}{x_i^0}\right)^2 + \frac{(x_i s_i^0)^2}{x_i s_i}\left(\frac{s_i^0 - \widetilde{s}_i}{s_i^0}\right)^2\right)^{1/2}.$$

Since $x_i s_i \ge (1-\beta)\tau$ and $\sum_{i=1}^{n}(s_i x_i^0)^2 + (x_i s_i^0)^2 \le (x^T s^0 + s^T x^0)^2$ we deduce that

$$\zeta(z, \tau\bar{b}) \le \frac{\sqrt{\tau}(x^T s^0 + s^T x^0)\phi(z^0)}{\tau_0\sqrt{1-\beta}},$$

and the statement of our proposition follows using Lemma 5 and the fact that $\mu/\tau \le 1 + \beta/\sqrt{2}$. □

We will now bound the norms of the vectors h^i appearing in (56).

Proposition 2 *If HLCP is $P_*(\kappa)$ and solvable then for any starting point $z^0 \in \mathcal{N}_2(\beta)$ there is a constant $\Lambda \ge 1$ (depending on the problem instance and the starting point z^0) such that for all iterations and all $m \ge 1$ the vectors h^i produced by the corrector step at each iteration of Algorithm 3 satisfy*

$$\frac{\|h^i\|_2}{\tau} \le \frac{2(1-\beta)\Lambda^i}{(1+2\kappa)i}, \quad i = m+1, \ldots, 2m. \tag{63}$$

Moreover, for any starting point z^0 satisfying (61) ($\chi = 0$), resp. (59) ($\chi = 1$), we have

$$\Lambda \le \lambda_\chi (1+\kappa)^{1+\chi} n^{(1+\chi)/2}, \quad \chi = 0, 1, \tag{64}$$

where the (universal) constants λ_0, λ_1 are given by

$$\lambda_0 = \frac{10(1+\sigma)}{1-\beta}, \qquad \lambda_1 = \frac{30(1+\sigma)}{1-\beta}\widehat{\gamma}, \tag{65}$$

with,

$$\widehat{\gamma} := \frac{1 + 1/\sqrt{2} + \max\{1, 2/\gamma\}}{3 + 1/\sqrt{2}} \max\left\{1, \left|\frac{1-\gamma}{\gamma}\right|\right\}. \tag{66}$$

Proof We start by using (56) and the notation $\bar{D} = \bar{X}^{-1/2}\bar{S}^{1/2}$ to obtain

$$\|h^i\|_2 \le \frac{1}{2}\sum_{j=i-m}^{m}\left(\|\bar{D}u^j\|_2\|\bar{D}^{-1}v^{i-j}\|_2 + \|\bar{D}u^{i-j}\|_2\|\bar{D}^{-1}v^j\|_2\right)$$
$$\le \frac{1}{2}\sum_{j=i-m}^{m}\|w^j\|_{\bar{z}}\|w^{i-j}\|_{\bar{z}}. \tag{67}$$

We want to find constants $\omega_1, \omega_2, \ldots, \omega_m$ such that

$$\|w^i\|_{\bar{z}} \leq \omega_i \sqrt{\tau}, \quad i = 1, 2, \ldots, m. \tag{68}$$

From Algorithm 3 it follows that $\bar{\tau} = \tau$ and $\lceil \bar{z}, \tau \rfloor \in \bar{\mathcal{N}}_2(\beta) \cap \mathcal{F}_{\bar{b}}$. Using (52) and Lemma 3 we deduce that

$$\|w^1\|_{\bar{z}} \leq \sqrt{1+2\kappa}\,\|\widetilde{a}^1\|_2 + (1+\sigma)(1+\sqrt{2+4\kappa})\zeta(\bar{z}, \tau\bar{b}), \tag{69}$$

$$\|w^2\|_{\bar{z}} \leq \sqrt{1+2\kappa}\,\|\widetilde{a}^2\|_2 + \sigma(1+\sqrt{2+4\kappa})\zeta(\bar{z}, \tau\bar{b}), \tag{70}$$

$$\|w^i\|_{\bar{z}} \leq \sqrt{1+2\kappa}\,\|\widetilde{a}^i\|_2, \quad i = 3, 4, \ldots, m, \tag{71}$$

where

$$\widetilde{a}^1 = -(1+\sigma)(\bar{x}\bar{s})^{1/2}, \qquad \widetilde{a}^2 = \sigma(\bar{x}\bar{s})^{1/2} - (\bar{x}\bar{s})^{-1/2}u^1v^1,$$

$$\widetilde{a}^i = -(\bar{x}\bar{s})^{-1/2}\sum_{j=1}^{i-1} u^j v^{i-j}, \quad i = 3, 4, \ldots, m.$$

In obtaining the following estimates we looked for simple, rather than tight, upper bounds

$$\|\widetilde{a}^1\|_2 = (1+\sigma)\sqrt{n\bar{\mu}} \leq (1+\sigma)\sqrt{(1+\beta/\sqrt{2})n\tau} < 2(1+\sigma)\sqrt{n\tau}, \tag{72}$$

$$\|\widetilde{a}^2\|_2 \leq 2\sigma\sqrt{n\tau} + \frac{\|u^1v^1\|_2}{\sqrt{(1-\beta)\tau}} \leq 2\sigma\sqrt{n\tau} + \frac{\|w^1\|_{\bar{z}}^2}{2\sqrt{(1-\beta)\tau}}, \tag{73}$$

$$\|\widetilde{a}^i\|_2 \leq \frac{1}{2\sqrt{(1-\beta)\tau}}\sum_{j=1}^{i-1}\|w^j\|_{\bar{z}}\|w^{i-j}\|_{\bar{z}}, \quad i = 3, 4, \ldots, m. \tag{74}$$

From (69), (70), (72), (73) and Proposition 1 (with $\bar{z}$ instead of z) it follows that (68) is satisfied for $i = 1, 2$, with

$$\omega_1 := (1+\sigma)\big(2\sqrt{(1+2\kappa)n} + \big(1+\sqrt{2+4\kappa}\,\big)\bar{\zeta}\big), \tag{75}$$

$$\omega_2 := \sqrt{1+2\kappa}\left(2\sigma\sqrt{n} + \frac{\omega_1^2}{2\sqrt{1-\beta}}\right) + \sigma\big(1+\sqrt{2+4\kappa}\,\big)\bar{\zeta}. \tag{76}$$

Let us define the (infinite) sequence

$$\omega_i := \frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}\sum_{j=1}^{i-1}\omega_j\omega_{i-j}, \quad i = 3, 4, \ldots. \tag{77}$$

Using (71) and (74), we can easily prove by induction that (68) is satisfied for $i = 1, 2, \ldots, m$. It is also easily proved by induction that

$$\omega_i \le \frac{\alpha_i}{c}(0.25\Lambda)^i \le \frac{\Lambda^i}{ci}, \quad i = 1, 2, \ldots, \tag{78}$$

where

$$c = \frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}, \quad \Lambda := 4\max\{c\omega_1, \sqrt{c\omega_2}, .25\}, \tag{79}$$

and (α_i) is the sequence (considered in [23])

$$\alpha_i := \frac{1}{i}\binom{2i-2}{i-1} \le \frac{1}{i}4^i,$$

which satisfies the following recurrence

$$\alpha_1 = 1, \qquad \alpha_i = \sum_{j=1}^{i-1} \alpha_j \alpha_{i-j}, \quad i = 2, 3, \ldots.$$

By construction, it follows that (78) holds for $i = 1, 2$. Using the recursions of the α_i's and the ω_i's, we deduce that (78) also holds for $i = 3, 4, \ldots$.

Let us consider now an index i, such that $m+1 \le i \le 2m$. Using (67), (68), and (78), we obtain

$$\begin{aligned} \|h^i\|_2 &\le \frac{1}{2}\sum_{j=i-m}^{m} \|w^j\|_z \|w^{i-j}\|_z \le \frac{\tau}{2}\sum_{j=i-m}^{m} \omega_j\omega_{j-i} \le \frac{\tau}{2}\sum_{j=1}^{i-1} \omega_j\omega_{j-i} \\ &\le \frac{\tau(0.25\Lambda)^i}{2c^2}\sum_{j=1}^{i-1}\alpha_j\alpha_{i-j} = \frac{\tau\alpha_i(0.25\Lambda)^i}{2c^2} \le \frac{\tau\Lambda^i}{2c^2 i} \le \tau\frac{2(1-\beta)}{1+2\kappa}\frac{\Lambda^i}{i}, \end{aligned}$$

which proves (63).

Let us assume that the starting point $z^0 = \lceil x^0, s^0 \rfloor$ satisfies condition (59). Since $Q(x^0 - x^*) + R(s^0 - s^*) = r^0$ it follows that

$$\phi(z^0) \le \|(z^0)^{-1}(z^0 - z^*)\|_\infty \le \bar{\gamma} := \max\left\{1, \left|\frac{1-\gamma}{\gamma}\right|\right\}, \qquad \zeta^* \le \frac{2}{\gamma},$$

$$\bar{\zeta} \le \frac{1+\beta/\sqrt{2}+\max\{1, 2/\gamma\}}{\sqrt{1-\beta}}\bar{\gamma}(1+4\kappa)n = \frac{\widehat{\gamma}(3+1/\sqrt{2})}{\sqrt{1-\beta}}(1+4\kappa)n.$$

Since the expression $(1+4\kappa)(1+\sqrt{2+4\kappa})(1+2\kappa)^{-3/2}$ attains its maximum for $\kappa = 7/4$, we deduce that

$$(1+4\kappa)\left(1+\sqrt{2+4\kappa}\right) \le \frac{64\sqrt{2}}{27}(1+2\kappa)^{3/2}, \quad \forall\kappa \ge 0.$$

It follows that for any starting point satisfying (59) we have

$$\widehat{\zeta} := (1+\sqrt{2+4\kappa})\bar{\zeta} < \frac{25\widehat{\gamma}}{2\sqrt{1-\beta}}(1+2\kappa)^{3/2}n.$$

Therefore, using (75), (76) and the majorization $\sqrt{n} \leq n/\sqrt{2}$, we obtain

$$\begin{aligned}
\frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}\omega_1 &\leq \frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}\left(1+\frac{2\sqrt{2}}{25}\right)\frac{25(1+\sigma)\widehat{\gamma}}{2\sqrt{1-\beta}}(1+2\kappa)^{3/2}n \\
&\leq \frac{7(1+\sigma)\widehat{\gamma}}{1-\beta}(1+2\kappa)^2 n, \\
\frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}\omega_2 &= \frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}\left(2\sigma\sqrt{n}+\frac{\omega_1^2}{2\sqrt{1-\beta}}\right)+\sigma\widehat{\zeta} \\
&\leq \frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}\left(1+\frac{\sigma}{98\sqrt{2}}\right)\frac{2(7(1+\sigma)\widehat{\gamma})^2}{\sqrt{1-\beta}}(1+2\kappa)^{7/2}n^3+\sigma\widehat{\zeta} \\
&\leq \left(1+\frac{\sigma}{98\sqrt{2}}+\frac{25\sigma}{392}\right)\frac{(7(1+\sigma)\widehat{\gamma})^2}{1-\beta}(1+2\kappa)^4 n^3 \\
&\leq \frac{53(1+\sigma)^2\widehat{\gamma}^2}{1-\beta}(1+2\kappa)^4 n^3.
\end{aligned}$$

It follows that if the starting point satisfies (59) then

$$\Lambda \leq \frac{30(1+\sigma)\widehat{\gamma}}{1-\beta}(1+2\kappa)^2 n^{3/2}.$$

Finally, if the starting point satisfies (61) we have

$$\begin{aligned}
\widehat{\zeta} &\leq 2\sqrt{(1+2\kappa)n}, \\
\frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}\omega_1 &\leq \frac{2(1+\sigma)}{1-\beta}(1+2\kappa)\sqrt{n}, \\
\frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}\omega_2 &= \frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}\left(2\sigma\sqrt{n}+\frac{\omega_1^2}{2\sqrt{1-\beta}}\right)+\sigma\widehat{\zeta} \\
&\leq \frac{\sqrt{1+2\kappa}}{2\sqrt{1-\beta}}\left(1+\frac{\sigma\sqrt{2}}{8}\right)\frac{8(1+\sigma)^2}{\sqrt{1-\beta}}(1+2\kappa)^{3/2}n+\sigma\widehat{\zeta} \\
&\leq \left(1+\frac{\sigma\sqrt{2}}{8}+\frac{\sigma\sqrt{2}}{4}\right)\frac{4(1+\sigma)^2}{1-\beta}(1+2\kappa)^2 n \\
&\leq \frac{6.2(1+\sigma)^2}{1-\beta}(1+2\kappa)^2 n,
\end{aligned}$$

which shows that in this case

$$\Lambda \leq \frac{10(1+\sigma)}{1-\beta}(1+2\kappa)\sqrt{n}. \qquad \square$$

We note if we take $\gamma = 1$ in the definition of $\widehat{\gamma}$ from (66) we obtain $\widehat{\gamma} = 1$.

In the remainder of this subsection we give explicit upper bounds for the number of iterations required by Algorithm 3 to obtain a solution of the HLCP with prescribed accuracy in case the starting point satisfies (59) or (61). More precisely given any $\varepsilon > 0$ we have to find an upper bound for the number

$$K_\varepsilon := \min\left\{K : \max\left\{x^{kT}s^k, \|r^k\|_2\right\} \leq \varepsilon, \forall k \geq K\right\}. \tag{80}$$

The upper bound will depend on n, κ, m,

$$L_\varepsilon := \log\max\left\{\frac{x^{0T}s^0}{\epsilon}, \frac{\|r^0\|_2}{\epsilon}\right\}, \tag{81}$$

and

$$\eta_\chi(\beta, m) := \frac{1}{2\lambda_\chi}\left(\frac{1}{2}\beta\left(1-\frac{\beta}{\sqrt{8}}\right)\right)^{\frac{1}{m+1}},$$

where λ_1, λ_2 are the constants from Proposition 2, and χ is given by (62). We note that $\eta_\chi(\beta, m)$ is increasing in m and that according to (64) we have $\eta_\chi(\beta, m) = O((1+\kappa)^{1+\chi}n^{\frac{1+\chi}{2}})$, which implies polynomial complexity for starting points satisfying (59) or (61).

Theorem 6 *If HLCP is sufficient and solvable then Algorithm 3 is globally convergent in the sense that for any starting point $z^0 \in \mathcal{N}_2(\beta)$ we have*

$$\lim_{k\to\infty} \tau_k = 0, \qquad \lim_{k\to\infty} \mu_k = 0, \qquad \lim_{k\to\infty} r^k = 0.$$

Moreover, if the starting point satisfies (59) or (61) then the number of iterates necessary to obtain an ε-approximate solution (80) can be bounded as follows

$$K_\varepsilon \leq \left\lceil \frac{1+\frac{\beta}{\sqrt{n}L_\varepsilon}}{\eta_\chi(\beta, m)}(1+\sqrt{8}\kappa)\sqrt{n}L_\varepsilon \right\rceil.$$

Proof At the beginning of a typical iteration of Algorithm 3 we have a triple $\lceil z, \tau \rfloor \in \bar{\mathcal{N}}_2(\beta)$. Since the corrector step of Algorithm 3 coincides with the corrector step of Algorithm 2, it follows from the proof of Theorem 4 that $\lceil \bar{z}, \tau \rfloor \in \bar{\mathcal{N}}_2(\bar{\beta})$ and

$$\beta - \bar{\beta} \geq \frac{(1-\beta)\beta(1-\frac{\beta}{\sqrt{8}})}{1+\sqrt{8}\kappa} > \frac{.7(1-\beta)\beta(1-\frac{\beta}{\sqrt{8}})}{1+2\kappa}.$$

In what follows, we will find a lower bound for the step length of the predictor defined in (53). To this effect we will use equation (57), Proposition 2, and the fact that

$$\sum_{i=m+1}^{2m} \frac{t^i}{i} \le t^{m+1} \sum_{i=m+1}^{2m} \frac{1}{i} < t^{m+1} \int_m^{2m} \frac{du}{u} = t^{m+1} \log 2 < 0.7, \quad \forall t \in (0, 1].$$

For any $\theta \in [0, \Lambda^{-1})$, where $\Lambda \ge 1$ is the constant whose existence was proved in Proposition 2, we have

$$\begin{aligned} \frac{\|\bar{x}(\theta)\bar{s}(\theta) - \bar{\tau}(\theta)e\|_2}{\bar{\tau}(\theta)} &\le \bar{\beta} + \frac{1}{(1-\theta)^{1+\sigma}\tau} \sum_{i=m+1}^{2m} \theta^i \|h^i\|_2 \\ &\le \bar{\beta} + \frac{2(1-\beta)}{(1+2\kappa)(1-\theta)^{1+\sigma}} \sum_{i=m+1}^{2m} \frac{(\theta\Lambda)^i}{i} \\ &\le \bar{\beta} + \frac{1.4(1-\beta)(\theta\Lambda)^{m+1}}{(1+2\kappa)(1-\theta)^{1+\sigma}} \\ &\le \beta - \frac{.7(1-\beta)}{1+2\kappa}\left(\beta(1-\beta/\sqrt{8}) - \frac{2(\theta\Lambda)^{m+1}}{(1-\theta)^{1+\sigma}}\right). \end{aligned} \tag{82}$$

Let us denote

$$\nu = \frac{m+1}{1+\sigma}, \quad a = \frac{2^{\frac{1}{1+\sigma}}\Lambda^{\nu}}{(\beta(1-\beta/\sqrt{8}))^{\frac{1}{1+\sigma}}}\cdot$$

It follows that

$$0 \le \theta < \Lambda^{-1} \quad \text{and} \quad \frac{\theta^{\nu}}{1-\theta} \le \frac{1}{a} \quad \Rightarrow \quad \lceil \bar{z}(\theta), \bar{\tau}(\theta) \rfloor \in \bar{\mathcal{N}}_2(\beta). \tag{83}$$

Using the fact that $a > 1$ it is easily checked that the function $g(\theta) := 1 - \theta - a\theta^{\nu}$ is decreasing on the interval (0, 1) and

$$g(0) = 1 > 0, \qquad g\left(\frac{1}{2}a^{-\frac{1}{\nu}}\right) = 1 - \frac{1}{2}a^{-\frac{1}{\nu}} - \frac{1}{2^{\nu}} > 0, \qquad g\left(a^{-\frac{1}{\nu}}\right) = -a^{-\frac{1}{\nu}} < 0.$$

It follows that $g(\theta)$ is positive on the interval $(0, .5a^{-\frac{1}{\nu}})$. According to (83) we have

$$\lceil \bar{z}(\theta), \bar{\tau}(\theta) \rfloor \in \bar{\mathcal{N}}_2(\beta), \quad \forall \theta \in [0, .5a^{-\frac{1}{\nu}}],$$

and using the definition (53) we deduce that

$$\theta^+ \ge \widetilde{\theta} = .5a^{-\frac{1}{\nu}} = \frac{(\beta(1-\beta/\sqrt{8}))^{\frac{1}{m+1}}}{2^{\frac{m+2}{m+1}}\Lambda}. \tag{84}$$

From Theorem 5 it follows that

$$\tau_k \le (1-\widetilde{\theta})^k \tau_0, \qquad \|r^k\|_2 \le (1-\widetilde{\theta})^k \|r^0\|_2, \qquad \mu_k \le (1+\beta/\sqrt{n})(1-\widetilde{\theta})^k \mu_0, \tag{85}$$

wherefrom we deduce that the sequences (τ_k), (μ_k), (r^k) converge to zero. This ends the first part of the theorem.

Let us assume now that the starting point satisfies (59) or (61). From (64) and (84) it follows that

$$\widetilde{\theta} \ge \frac{1}{2\lambda_\chi}\left(\frac{1}{2}\beta\left(1-\frac{\beta}{\sqrt{8}}\right)\right)^{\frac{1}{m+1}} ((1+\kappa)\sqrt{n})^{-(1+\chi)} = \frac{\eta_\chi(\beta,m)}{(1+\kappa)^{1+\chi} n^{\frac{1+\chi}{2}}}$$

and the bound is obtained as in the proof of Corollary 1. □

From the above theorem it follows that Algorithm 3 has $O((1+\kappa)^2 nL)$-iteration complexity for starting points satisfying (59) and $O((1+\kappa)\sqrt{n}L)$-iteration complexity for starting points satisfying (61). We note that the polynomial complexity does not depend on the values of the parameters σ and m defining our algorithm. In the next section we will see that the asymptotic rate of convergence does depend on those parameters.

4.3 Superlinear convergence

In this subsection we will prove that Algorithm 3 is superlinearly convergent for general problems, provided σ satisfies (10) and $(\sigma, m) \ne (1,1)$. More precisely we will show that the Q-order of convergence of Algorithm 3 is at least $\frac{m+1}{\sigma+1}$. The proof of superlinear convergence is based on the following lemma

Lemma 6 *If HLCP is sufficient and solvable then there is a positive constant ω such that the vectors $u^1, v^1, \ldots, u^m, v^m$ computed in the predictor step at each iteration of the Algorithm 3 satisfy*

$$\|u^i\|_2 \le \sqrt{\omega}\tau^{\frac{i}{1+\sigma}}, \quad \|v^i\|_2 \le \sqrt{\omega}\tau^{\frac{i}{1+\sigma}}, \quad i = 1, 2, \ldots, m.$$

Proof Since for all the points $\bar{x}, \bar{s}$ produced by the algorithm we have $(1-\beta)e \le \bar{x}\bar{s}/\tau \le (1+\beta)e$, it follows that the set $\mathcal{K}$ of all the points $\bar{x}\bar{s}/\tau$ produced by the algorithm form a compact subset of $\mathbb{R}^n_{++}$. The desired results is obtained by applying Theorem 2. □

With the help of the lemma above we obtain the following result.

Theorem 7 *If HLCP is sufficient and solvable and $(\sigma, m) \ne (1,1)$, then sequences of the complementarity gaps (μ_k), feasibility measures (τ_k), and residuals (r_i^k), $i = 1, \ldots, n$, produced by Algorithm 3, converge Q-superlinearly to zero. Moreover, the Q-orders of convergence of these sequences satisfy*

$$\mathcal{Q}(\mu_k) = \mathcal{Q}(\tau_k) = \mathcal{Q}(r_i^k) \ge \frac{m+1}{1+\sigma}. \tag{86}$$

Proof From Lemma 6 it follows that

$$\left\|h^i\right\|_2 \leq (2m+1-i)\omega\tau^{\frac{i}{1+\sigma}} \leq m\omega\tau^{\frac{i}{1+\sigma}}, \quad i=m+1,\dots,2m.$$

Since $\lim \tau_k = 0$, we may assume without loss of generality that

$$\tau^{\frac{1}{1+\sigma}} < \frac{1}{2}, \qquad \frac{2m\omega}{\beta-\bar{\beta}}\tau^{\frac{m-\sigma}{1+\sigma}} < 1.$$

Proceeding as in (82), and using the formula for the sum of a geometric series with ration $1/2$, we deduce that for any $\theta \in (0,1)$ we have

$$\frac{\|\bar{x}(\theta)\bar{s}(\theta)-\bar{\tau}(\theta)e\|_2}{\bar{\tau}(\theta)} \leq \bar{\beta} + \frac{m\omega}{(1-\theta)^{1+\sigma}\tau}\sum_{i=m+1}^{2m}\tau^{\frac{i}{1+\sigma}} < \psi_\tau(\theta) := \bar{\beta} + \frac{2m\omega\tau^{\frac{m+1}{1+\sigma}}}{(1-\theta)^{1+\sigma}\tau}.$$

It follows that

$$0 < \theta \leq \widetilde{\theta}_\tau := 1 - \left(\frac{2m\omega}{\beta-\bar{\beta}}\tau^{\frac{m-\sigma}{1+\sigma}}\right)^{\frac{1}{1+\sigma}} \quad \Rightarrow \quad \lceil \bar{z}(\theta), \bar{\tau}(\theta) \rfloor \in \bar{\mathcal{N}}_2(\beta),$$

which according to (53) implies $\theta^+ \geq \widetilde{\theta}_\tau$. Hence

$$\tau^+ = (1-\theta^+)^{1+\sigma}\tau \leq \frac{2m\omega}{\beta-\bar{\beta}}\tau^{\frac{m+1}{1+\sigma}},$$

$$\mu^+ \leq (1+\beta/\sqrt{2})\tau^+ \leq \frac{(2+\beta\sqrt{2})m\omega}{(\beta-\bar{\beta})(1-\beta/\sqrt{2})^{\frac{m+1}{1+\sigma}}}\mu^{\frac{m+1}{1+\sigma}},$$

which proves that the sequences (τ_k) and (μ_k) converge to zero with Q-order at least $\frac{m+1}{1+\sigma}$. If the i'th residual at the starting point, r_i^0, does not vanish then,

$$|r_i^+| = \frac{|r_i^0|\tau^+}{\tau_0} \leq \frac{2m\omega|r_i^0|}{(\beta-\bar{\beta})\tau_0}\tau^{\frac{m+1}{1+\sigma}} \leq \frac{2m\omega\tau_0^{\frac{m-\sigma}{1+\sigma}}}{(\beta-\bar{\beta})|r_i^0|^{\frac{m-\sigma}{1+\sigma}}}|r_i|^{\frac{m+1}{1+\sigma}}.$$

The proof is complete. □

5 Numerical experiments

The purpose of this section is to test numerically the theoretical results obtained in this paper. In the first part of the section we will give numerical results for some simple $P_*(\kappa)$ LCPs having one of the following properties:

P1. The LCP has a unique strictly complementary solution;
P2. The solution set of the LCP is bounded, does not reduce to a point, and contains at least one strictly complementary solution;
P3. The LCP has a unique solution that is not strictly complementary;

P4. The solution set of the LCP is bounded, does not reduce to a point, and does not contain any strictly complementary solution;
P5. The solution set of the LCP is not bounded;

We first prove the following simple result.

Lemma 7 *The matrix*

$$Q_2 = \begin{pmatrix} 0 & 1+4\kappa \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

is $P_*(\kappa)$, $\forall \kappa \geq 0$.

Proof We will verify (3). Since $x(Q_2x) = \lceil 1+4\kappa)x_1x_2, -x_1x_2\rfloor$, the following situations may occur:

1. $x_1x_2 > 0$, in which case $\hat{\mathcal{I}}^+ = \{1\}$, $\hat{\mathcal{I}}^- = \{2\}$, and

$$(1+4\kappa)\sum_{i\in\hat{\mathcal{I}}^+} x_i[Q_2x]_i + \sum_{i\in\hat{\mathcal{I}}^-} x_i[Q_2x]_i = ((1+4\kappa)^2-1)x_1x_2 \geq 0.$$

2. $x_1x_2 < 0$, in which case $\hat{\mathcal{I}}^+ = \{2\}$, $\hat{\mathcal{I}}^- = \{1\}$, and

$$(1+4\kappa)\sum_{i\in\hat{\mathcal{I}}^+} x_i[Q_2x]_i + \sum_{i\in\hat{\mathcal{I}}^-} x_i[Q_2x]_i = (1+4\kappa)(-x_1x_2) + (1+4\kappa)x_1x_2 = 0.$$

□

The following lemma shows that one can obtain a $P_*(\kappa)$ LCP having properties P1, P2, and P5 by using the matrix Q_2 from Lemma 7 in conjunction with an appropriate $b = \lceil b_1, b_2\rfloor \in \mathbb{R}^2$.

Lemma 8 *The* $P_*(\kappa)$ *LCP*

$$xs = 0, \quad s = Q_2x - b, \quad x \geq 0, s \geq 0,$$

has

- *no solution when* $b_2 > 0$;
- *an unbounded solution set when* $b_2 = 0$;
- *a unique solution when* $b_1 \neq 0$ *and* $b_2 < 0$;
- *a bounded infinite solution set when* $b_1 = 0$ *and* $b_2 < 0$.

Proof The solution of our $P_*(\kappa)$ LCP satisfies

$$s_1 = (1+4\kappa)x_2 - b_1 \geq 0, \quad s_2 = -x_1 - b_2 \geq 0,$$
$$x_1s_1 = x_2s_2 = 0, \quad x_1 \geq 0, x_2 \geq 0.$$

If $b_2 > 0$, then $s_2 = -x_1 - b_2 < 0$, so that the LCP has no solution.

If $b_2 = 0$, then $s_2 = -x_1$, which implies $x_1 = s_2 = 0$. By taking $x_2 = \beta$ and $s_1 = (1+4\kappa)\beta - b_1$, we obtain a solution of the LCP for any $\beta \geq \max\{b_1/(1+4\kappa), 0\}$, which shows that the solution set in unbounded.

If $b_2 < 0$ and $x_1 = 0$, then $s_2 = -b_2 > 0$. Therefore we must have $x_2 = 0$ and $s_1 = -b_1$, which gives a solution whenever $b_1 \leq 0$. If $b_2 < 0$ and $x_1 > 0$ then $s_1 = 0$, so that we must have $x_2 = b_1/(1 + 4\kappa)$, which is a valid solution as long as $b_1 \geq 0$. If $b_2 < 0$ and $b_1 > 0$, then we must have $s_2 = 0$ (and therefore $x_1 = -b_2$), while if $b_2 < 0$ and $b_1 = 0$, then by taking $x_1 = \beta$, $s_2 = -\beta - b_2$, we obtain a solution for any $\beta \in [0, -b_2]$. □

We will construct a $P_*(\kappa)$-LCP that does not have any strictly complementary solutions using the matrix Q_3 considered in the following lemma:

Lemma 9 *The matrix*

$$Q_3 = \begin{pmatrix} 0 & 1+4\kappa & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

is $P_*(\kappa)$, $\forall \kappa \geq 0$.

Proof Since $x(Qx) = \lceil (1+4\kappa)x_1x_2, -x_1x_2, x_3^2 \rfloor$, it follows that $3 \in \hat{\mathcal{I}}^+$. Therefore we can proceed as in the proof of Lemma 7 to show that (3) is satisfied. We note that the extra term $(1+4\kappa)x_3^2$ appearing in the summation keeps the right-hand side non-negative. □

The following lemma shows how to choose $b = \lceil b_1, b_2, b_3 \rfloor \in \mathbb{R}^3$, such that LCP with matrix Q_3 has the property P3 or P4.

Lemma 10 *The* $P_*(\kappa)$ *LCP*

$$xs = 0, \quad s = Q_3x - b, \quad x \geq 0, \ s \geq 0,$$

has:

- *a unique solution that is not strictly complementary when* $b_1 \neq 0, b_3 = 0$, *and* $b_2 < 0$;
- *a bounded infinite solution set that does not contain any strictly complementary solution when* $b_1 = b_3 = 0$ *and* $b_2 < 0$.

Proof The solution of the LCP satisfies

$$s_1 = (1+4\kappa)x_2 - b_1 \geq 0, \quad s_2 = -x_1 - b_2 \geq 0, \quad s_3 = x_3 - b_3 \geq 0, \quad x \geq 0,$$
$$xs = 0.$$

Remark that $b_3 = 0$ implies $s_3 = x_3 = 0$. Thus for $b_3 = 0$ there are no strictly complementary solutions. Since the first 2×2 block of Q_3 coincides with Q_2 we can use Lemma 8 to complete the proof. □

For our numerical examples we will construct $P_*(\kappa)$ matrices of any size as block diagonal matrices having the matrices Q_2 and Q_3 on the diagonal. We first apply our algorithms to $P_*(\kappa)$ LCPs that use the matrices Q_2 and Q_3 defined in Lemmas 7 and 9. We obtain LCPs whose solution sets have one of the properties P1–P5 by choosing appropriately the components of b based on Lemmas 8 and 10. The stopping criteria consist in $\mu_k \le 10^{-8}$ and $r_k \le 10^{-8}$, where $\mu_k = (x_k^T s_k)/n$ is the duality gap and $r^{k+1} = Qx_k + s_k - b$ is the residual at current iteration.

Table 1 contains the number of iterations needed by the algorithm with various orders ($m = 1$, $m = 2$, $m = 3$ and $m = 4$) to converge. The matrix Q of the LCP is block diagonal with a succession of Q_2 and Q_3 matrices on the diagonal. Since in this case the linear equalities are decoupled, we can choose the components of the vector b according to Lemmas 8 and 10 in order to obtain an LCP having one of the properties P1–P5. We were also interested on how the same order algorithms perform under the assumption that the $P_*(\kappa)$ has ($\sigma = 0$) or has not ($\sigma = 1$) a strictly complementary solution. The results illustrate that, as predicted by the theoretical analysis, the algorithms using $\sigma = 0$ perform better when the problems are nondegenerate (P1, P2 and P5) while the algorithms using $\sigma = 1$ perform better for degenerate problem (P3 and P4).

The numerical results presented in Table 1, while confirming the theory, are done on some very simple and artificial examples. Since to our knowledge there is no collection of benchmark problems for sufficient linear complementarity problems, we decided to present in the second part of this section the results obtained by running our algorithm on the linear programming problems form the NETLIB LP Repository (http://www.netlib.org/lp/data). We compare our algorithms with the predictor-corrector method for linear programming acting in $\mathcal{N}_2(0.5)$ described in [11]. Since this algorithm needs to start at an initial strictly feasible point, we adapted the technique from [22] to our general formulation of the linear programs and obtained a homogeneous self-dual LP (HSD LP) model for which a strictly feasible starting point is known. The algorithm stops when duality gap is less than 10^{-12}. Since any linear programming problem has strictly complementary solution we used our algorithms with $\sigma = 0$.

In our first numerical test we solved the HSD LPs using the MTY predictor-corrector as described in [11] and our first order corrector-predictor method (Algorithm 2). By contrast with the MTY algorithm that uses two fixed neighborhoods ($\mathcal{N}_2(.25)$ and $\mathcal{N}_2(.5)$), Algorithm 2 can work in any $\mathcal{N}_2(\beta)$ neighbourhood, with $\beta \in (0, 1)$. In our numerical experiments we used $\beta = 0.5$ and $\beta = 0.99$. The results are shown in Table 2. The second column n specifies the size of the HSP LP. The column *ite* specify the number of iterations needed by each method to solve the problem. The columns μ and $\|r\|$ represent the duality gap and the norm of the residual at the solution, respectively. As we expected, MTY requires the same number of iterations as our algorithm with $\beta = 0.5$ to obtain a solution within the same accuracy. The number of iterations obtained using $\beta = 0.99$ shows that a wider neighborhood ($\mathcal{N}_2(0.99)$) causes an improved behavior of the algorithm.

In the second numerical experiment we solved the same problems as before using Algorithm 3 with $\beta = 0.99$ and $m = 2, 3, 4$. The columns of Table 3 have the same meaning as those of Table 2. We note that the number of corrector-predictor steps needed for obtaining the same accuracy decreases as the order of the method increases. The use of higher order methods only slightly increases the computational

Table 1 Number of iterations: $n = 300$, $m \in \{1, 2, 3, 4\}$, $\sigma \in \{0, 1\}$

Type	κ	m, σ						
		1, 0	2, 0	2, 1	3, 0	3, 1	4, 0	4, 1
P1	0	30	19	19	14	14	13	12
	1	36	21	24	17	17	14	15
	100	84	56	59	49	48	45	46
	1000	150	111	115	96	98	92	92
	10000	188	150	151	128	132	125	125
P2	0	23	14	14	11	11	9	9
	1	23	13	16	11	11	9	10
	100	21	12	14	10	11	8	9
	1000	22	13	16	10	11	9	9
	10000	22	13	16	10	12	9	9
P3	0	41	25	16	19	12	16	10
	1	50	29	24	23	18	19	15
	100	80	52	52	45	42	39	38
	1000	123	90	90	78	76	75	72
	10000	173	138	135	121	118	116	111
P4	0	41	24	13	19	9	16	8
	1	46	27	17	22	12	18	10
	100	37	22	14	18	10	15	9
	1000	38	23	15	18	11	16	9
	10000	38	23	16	18	11	16	9
P5	0	13	7	10	6	8	5	5
	1	11	5	9	4	7	4	5
	100	7	4	7	4	5	3	4
	1000	7	4	7	3	5	3	4
	10000	7	5	7	4	6	3	4

cost per iteration. All versions discussed in this article use two matrix factorizations per iteration, one in the predictor step and one in the corrector step. In the case of dense matrices a factorization requires $O(n^3)$ arithmetic operations. Our mth order algorithm requires an additional m backsolves and forward solves per iteration, the cost of each being $O(n^2)$ arithmetic operations in case of dense factors. Hence, for large problems, where the number of iterations is much less than the size n, we advocate for the use of higher order methods.

Table 2 and especially the data for higher order versions from Table 3 show that the duality gap decreased from a value greater than 10^{-12} to a value of 10^{-14}, 10^{-15} and even less, during the last iteration. The gain of several digits of accuracy in one iteration is due to the high order of convergence of our methods.

Table 2 Performance of MTY and Algorithm 2 with $\beta = 0.5$ and $\beta = 0.99$ on Netlib LP problems

Problem	n	MTY			Algorithm 2					
					$\beta = 0.50$			$\beta = 0.99$		
		ite	μ	$\|r\|$	ite	μ	$\|r\|$	ite	μ	$\|r\|$
AGG3	759	56	8e–15	3e–09	56	8e–14	7e–09	41	2e–14	1e–08
AGG	616	56	6e–15	7e–09	56	2e–13	4e–09	41	3e–14	2e–08
BLEND	115	25	5e–15	4e–14	25	2e–13	1e–13	19	1e–14	1e–13
ETAMACRO	1034	91	8e–13	2e–11	92	1e–12	1e–11	66	8e–13	2e–11
E226	473	51	4e–14	5e–13	52	1e–15	8e–13	38	2e–15	1e–12
FFFFF800	1029	90	5e–13	4e–09	91	6e–13	3e–09	65	1e–12	5e–09
FINNIS	1146	80	7e–13	2e–11	81	6e–13	2e–11	58	8e–13	1e–11
FIT1D	2076	56	5e–16	4e–10	56	3e–14	5e–10	41	6e–16	4e–10
FIT1P	2077	60	1e–13	4e–10	61	1e–15	3e–10	44	2e–13	8e–10
GANGES	2104	78	5e–13	7e–10	79	1e–13	7e–10	57	7e–14	2e–09
GFRD-PNC	1419	52	9e–13	2e–10	53	3e–13	1e–10	38	2e–13	3e–10
GROW15	1246	59	9e–14	4e–08	59	5e–13	1e–08	43	1e–13	2e–08
GROW22	1827	65	6e–14	8e–08	65	4e–13	4e–08	47	7e–13	2e–08
GROW7	582	50	1e–14	1e–08	50	2e–13	3e–08	37	3e–15	1e–08
ISRAEL	317	57	4e–13	6e–10	58	3e–13	1e–09	42	1e–13	8e–10
KB2	78	38	6e–17	2e–13	38	3e–14	2e–13	28	5e–14	2e–13
LOTFI	367	55	2e–13	4e–11	56	9e–15	4e–11	40	7e–13	3e–11
MAROS	2002	88	8e–13	9e–11	89	8e–13	1e–10	64	8e–13	1e–10
RECIPE	300	34	5e–15	2e–12	34	8e–15	5e–12	25	2e–14	3e–12
SCAGR25	672	48	2e–17	2e–10	48	1e–15	1e–10	35	2e–14	3e–10
SCAGR7	186	35	7e–16	3e–11	35	3e–14	2e–11	26	2e–14	2e–11
SCFXM1	601	60	4e–13	4e–12	61	5e–14	4e–12	45	3e–16	7e–12
SCFXM2	1201	74	4e–15	2e–11	75	2e–15	1e–11	54	4e–15	1e–11
SCFXM3	1801	81	1e–14	4e–11	82	2e–15	4e–11	59	6e–14	3e–11
SCRS8	1276	67	7e–14	1e–11	68	9e–15	6e–12	49	6e–13	5e–11
SCTAP1	661	48	1e–13	1e–12	49	6e–15	3e–12	36	2e–15	5e–12
SCTAP2	2501	41	3e–14	3e–11	41	3e–14	3e–11	30	3e–13	2e–11
SHARE1B	254	71	1e–13	3e–12	72	2e–13	5e–12	52	2e–13	4e–12
SHARE2B	163	28	2e–13	5e–13	29	2e–14	9e–13	21	4e–13	4e–13
STAIR	697	50	1e–13	2e–12	50	3e–13	3e–12	37	1e–13	2e–12
STANDATA	1395	43	6e–15	1e–11	44	3e–15	2e–12	32	7e–15	6e–12
STANDMPS	1395	55	2e–14	2e–11	56	7e–15	2e–11	41	4e–15	2e–11
STOCFOR1	166	37	2e–16	7e–13	37	7e–15	1e–12	28	7e–17	9e–13
STOCFOR2	3046	97	3e–14	8e–12	98	5e–16	1e–11	71	8e–17	1e–11
TUFF	656	66	7e–13	6e–12	67	5e–13	9e–12	48	5e–13	1e–11
VTP.BASE	429	56	4e–13	1e–10	57	4e–13	9e–11	41	8e–13	1e–10
WOOD1P	2596	45	1e–14	8e–09	45	4e–13	1e–09	33	2e–14	6e–11

Table 3 Performance of higher order versions of our algorithm ($\beta = 0.99$) on NETLIB LP problems

Problem	n	$m = 2$			$m = 3$			$m = 4$		
		ite	μ	$\\|r\\|$	ite	μ	$\\|r\\|$	ite	μ	$\\|r\\|$
AGG3	759	25	3e–14	2e–11	21	4e–15	3e–09	19	7e–18	5e–11
AGG	616	25	2e–16	1e–11	20	7e–13	3e–08	18	5e–13	6e–12
BLEND	115	13	3e–17	1e–14	11	1e–16	1e–13	9	5e–15	2e–14
ETAMACRO	1034	45	1e–13	3e–13	37	3e–14	4e–11	34	1e–13	3e–13
E226	473	24	5e–16	4e–13	20	2e–15	2e–12	18	3e–17	3e–13
FFFFF800	1029	40	7e–13	2e–12	34	4e–15	7e–09	30	4e–13	8e–12
FINNIS	1146	38	2e–14	2e–11	32	7e–13	4e–11	29	7e–15	1e–11
FIT1D	2076	25	7e–13	8e–12	21	8e–20	4e–10	19	3e–15	2e–11
FIT1P	2077	27	2e–20	5e–12	22	4e–15	7e–10	19	2e–13	3e–12
GANGES	2104	32	5e–15	5e–12	25	2e–14	1e–09	21	9e–13	5e–13
GFRD-PNC	1419	23	3e–17	4e–11	18	3e–13	3e–10	16	3e–13	5e–11
GROW15	1246	25	2e–14	1e–12	20	1e–15	3e–08	17	2e–15	1e–12
GROW22	1827	27	1e–14	2e–12	21	1e–15	3e–07	18	1e–14	5e–12
GROW7	582	22	1e–13	3e–13	18	6e–16	1e–08	16	9e–16	8e–13
ISRAEL	317	28	2e–13	2e–13	23	6e–14	3e–09	21	1e–15	1e–13
KB2	78	19	1e–18	2e–14	16	9e–15	2e–13	14	4e–15	1e–14
LOTFI	367	25	7e–13	2e–12	21	1e–14	4e–11	18	6e–18	1e–11
MAROS	2002	39	3e–16	1e–11	32	5e–13	1e–10	28	2e–15	5e–12
RECIPE	300	16	7e–14	6e–14	13	7e–15	1e–11	11	1e–13	6e–14
SCAGR25	672	21	8e–14	1e–11	17	5e–17	4e–10	15	1e–13	2e–11
SCAGR7	186	17	2e–17	2e–12	14	1e–14	2e–11	13	3e–20	3e–12
SCFXM1	601	28	2e–13	6e–13	24	5e–13	6e–12	22	2e–18	2e–13
SCFXM2	1201	34	3e–14	9e–13	29	1e–15	3e–11	26	1e–17	2e–13
SCFXM3	1801	36	6e–16	4e–12	30	1e–15	2e–11	27	2e–14	3e–12
SCRS8	1276	31	7e–14	3e–12	26	1e–14	3e–11	24	9e–17	4e–12
SCTAP1	661	24	6e–16	1e–12	21	2e–15	4e–12	19	9e–16	2e–12
SCTAP2	2501	17	2e–15	1e–11	13	3e–14	3e–11	12	2e–15	2e–11
SHARE1B	254	35	2e–16	9e–14	29	7e–19	6e–12	26	1e–14	7e–14
SHARE2B	163	14	2e–17	1e–13	11	1e–14	6e–13	10	4e–13	2e–13
STAIR	697	22	3e–13	7e–13	18	2e–13	1e–12	16	1e–15	1e–12
STANDATA	1395	21	3e–16	5e–13	16	5e–13	9e–12	15	9e–13	2e–12
STANDMPS	1395	26	1e–15	7e–13	21	4e–14	2e–11	19	5e–15	2e–12
STOCFOR1	166	17	5e–15	3e–14	14	3e–16	1e–12	13	6e–17	7e–14
STOCFOR2	3046	41	1e–14	5e–12	33	1e–16	4e–11	30	1e–15	3e–11
TUFF	656	28	6e–13	6e–13	23	3e–13	7e–12	20	2e–13	8e–13
VTP.BASE	429	28	5e–17	3e–12	23	3e–13	8e–11	22	9e–17	1e–11
WOOD1P	2596	21	6e–15	4e–12	18	8e–14	1e–08	15	9e–13	4e–12

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