Discrete time Markov chains

Muruhan Rathinam

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1 Introduction

We introduce some concepts and definitions here.

A **stochastic process** is a family of random variables $\{X_t, t \in \mathcal{I}\}$. Here the "index variable" t takes values in the "index set" $\mathcal{I}$. Typically t is interpreted as time. The set $\mathcal{I}$ then represents the possible values of time under consideration. Typically we are interested in the cases where $\mathcal{I} = \{t \in \mathbb{R}, t \geq 0\}$ (continuous time processes) or $\mathcal{I} = \{0, 1, 2, \dots\}$ (discrete time processes). Here we shall only consider the discrete time case and it is customary to use n or k etc., instead of t for the index variable (or time variable).

The **state space** $\mathcal{S}$ of a stochastic process $\{X_n, n \geq 0\}$ is the set of all possible values that any X_n can take. Here we shall only consider state spaces that are "discrete", but possibly infinite. Examples are

$$\mathcal{S} = \{1, 2, \dots\},$$
$$\mathcal{S} = \{1, 2, 3, \dots, M\},$$
$$\mathcal{S} = \{A, B, C, D\},$$
$$\mathcal{S} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\},$$
$$\mathcal{S} = \{(i, j) | i \geq 0, j \geq 0\}.$$

The first example is all positive integers, an infinite set. The second and third examples are finite sets. The last example is the two dimensional lattice of nonnegative integers.

We shall be concerned with a special kind of stochastic process called the **Markov chain**.

Definition 1 *The stochastic process* $\{X_n, n \geq 0\}$ *with state space* $\mathcal{S}$ *is called a* **Markov chain** *(or a* **Markov process***) if*

$$\begin{aligned} &P\{X_n = j | X_{n-1} = i, X_{n-2} = i_2, \ldots, X_2 = i_{n-2}, X_1 = i_{n-1}, X_0 = i_n\} \\ &\quad = P\{X_n = j | X_{n-1} = i\}, \quad \textit{for all } j, i, i_2, \ldots, i_n \in \mathcal{S}. \end{aligned} \tag{1}$$

The equation (1) *is called the* **Markov property**. *The conditional probabilities* $P\{X_n = j | X_{n-1} = i\}$ *are called the* **state transition probabilities** *or simply the* **transition probabilities**.

In words, the state of the process at time n, given the knowledge of the state of the process at time $n-1$ is independent of any information about the states prior to time $n-1$.

Definition 2 *The Markov chain* $\{X_n, n \geq 0\}$ *is said to be* **homogeneous** *or* **time independent** *if the transition probabilities* $P\{X_n = j | X_{n-1} = i\}$ *do not depend on time* n. *In this case we may write*

$$P\{X_n = j | X_{n-1} = i\} = P_{ij}. \tag{2}$$

Since for any discrete state space $\mathcal{S}$ *we may relabel the states by integers* $1, 2, \ldots$ *we may regard* P_{ij} *as forming a matrix. (Some prefer to label them starting from* 0*, which is ok too!). The matrix* P *with entrees* P_{ij} *is called the* **(state) transition matrix**.

We make following observations.

- It must be observed that the dynamics of a homogeneous Markov chain are completely characterized by P_{ij}. It is also easy to see that

  $$\sum_j P_{ij} = 1.$$

  i.e. **The row sums must equal 1 always**.

- Note that when the state space $\mathcal{S}$ is finite, the matrix P is $M \times M$ where M is the number of states.

Remark 3 *In the rest of this discussion we shall only be concerned with homogeneous Markov chains.*

2 Dynamic equations : Chapman-Kolmogorov

A Markov chain with a given transition matrix P describes a stochastic dynamics where a system takes different states at different times depending on where it was at the previous time in a random manner. The value P_{ij} is the probability that the system will move from state i to state j after one time step. If we are given the **probability mass function (pmf)** $p_n(i), i = 0, 1, \ldots$ for X_n then we can compute the pmf $p_{n+1}(i), i = 0, 1, \ldots$ for X_{n+1} by the formula

$$p_{n+1}(j) = \sum_i P_{ij} p_n(i)$$

which may be written as the row-vector matrix product

$$p_{n+1} = p_n P$$

where p_{n+1} and p_n are row vectors giving the corresponding pmfs.

It follows by induction that if we know the pmf p_n at time n, i.e. the pmf for X_n then we obtain the pmf at any later time $n + k$ by the equation

$$p_{n+k} = p_n P^k. \tag{3}$$

These are known as the **Chapman-Kolmogorov** equations and may be also written

$$P\{X_{n+k} = j | X_n = i\} = (P^k)_{ij}.$$

Remark 4 *In words this means the conditional probability of being in state state j after k transitions starting in state i is given by the (i, j)th entry of the matrix P^k.*

Also note that if the pmf for initial state X_0 is given to be p_0 then the pmf for any state n is given by

$$p_n = p_0 P^n.$$

3 Classification of states into classes

For a Markov chain with a discrete state space $\mathcal{S}$ the various states may be classified according to various criteria of how they communicate with each other. This classification often enables us to come to strong conclusions about the long term behaviour of the chain (process). Some important terms and concepts that come up in this context are **reachable, communicate, absorbing, classes, irreducible, closed**. We shall define and discuss these terms.

Definition 5 *A state j is* **reachable** *from state i if there exists $k \geq 0$ such that*

$$P\{X_{n+k} = j | X_n = i\} = (P^k)_{ij} > 0.$$

If state j is reachable from i, we write it as $i \to j$.

In words this means that it is possible (i.e. there is a nonzero probability) that the chain could visit state j after having visited i.

Remark 6 *Note that by this definition state i is reachable from state i (i.e. $i \to i$), since we included the case $k = 0$ in the definition. If we had used $k > 0$ in our definition then it may not be the case, but this definition ($k \geq 0$) turns out to be useful in the long run.*

It follows from the definition that

1. $i \to i$ for all i (see remark above).
2. If $i \to j$ and $j \to k$ then it follows that $i \to k$.

Definition 7 *A state i is said to be* **absorbing** *if there exists no state j distinct from i such that $i \to j$.*

In words this means if you are in state i at any time, then (with probability one) you will remain in state i for ever.

Remark 8 *An absorbing state can be identified from looking at the main diagonal of P. A diagonal entry $P_{ii} = 1$ if and only if i is an absorbing state.*

Definition 9 *We say states i and j* **communicate** *if and only if $i \to j$ and $j \to i$. We write it as $i \leftrightarrow j$.*

It follows from the definition that

1. $i \leftrightarrow i$ for all i.
2. If $i \leftrightarrow j$ then $j \leftrightarrow i$.

3. If $i \leftrightarrow j$ and $j \leftrightarrow k$, then it follows that $i \leftrightarrow k$.

For those who are familiar with the concept of an **equivalence relation** it is worth to note that $\leftrightarrow$ is an equivalence relation. It follows from above properties that the state space can be partitioned into mutually disjoint classes $C_1, C_2, \ldots$, where elements (states) of any given class communicate with each other and not with any elements of any other class. Note that $\mathcal{S}$ is the disjoint union of these classes. It is worth noticing that any absorbing state forms a class by itself.

Definition 10 *A Markov chain which consists of a single class is called* **irreducible**.

Just means that it can not be "reduced" to a number of disjoint classes!

Definition 11 *A class C is said to be* **closed** *if any state $j \notin C$ can not be reached from any state $i \in C$.*

In words, if the process starts from class C or if it ever enters class C then it will remain there for ever. Note that it is **possible to enter but not leave** a closed class. Absorbing states are special cases of closed classes. Any absorbing state i forms a single element class $\{i\}$ that is closed.

The idea of splitting the state space into classes is very useful because several interesting properties of states that relate to the long term behaviour are usually shared by all elements of a class. In other words either all the states of a class have that property or none of them have that property. Such a property is called a **class property**. Suppose $\mathcal{A}$ is a property and we say $\mathcal{A}(i)$ is true or false depending on whether state i has property $\mathcal{A}$. If $\mathcal{A}$ is a class property and suppose state $i \in C$ (C being a class)then $\mathcal{A}(i)$ is true if and only if $\mathcal{A}(j)$ is true for all states $j \in C$.

4 Recurrence and transience

The concepts of recurrence and transience play a critical role in the study of the long term behaviour of a Markov chain. Every state i of a Markov chain can be categorized as either **recurrent** or **transient** according to the following criterion.

First let us define the *indicator* random variables $I_n(i)$ for $i \in \mathcal{S}$ and $n = 0, 1, 2, \ldots$ by $I_n(i) = 1$ if $X_n = i$ and $I_n(i) = 0$ if $X_n \neq i$. Define V_i as the number of times state i is visited by the process during times $n = 1, 2, \ldots$. Thus

$$V_i = \sum_{n=1}^{\infty} I_n(i),$$

with the possibility that $V_i = \infty$. It follows that

$$\begin{aligned} E(V_i \mid X_0 = i) &= \sum_{n=1}^{\infty} E(I_n(i) \mid X_0 = i) \\ &= \sum_{n=1}^{\infty} P\{X_n = i \mid X_0 = i\} = \sum_{n=1}^{\infty} (P^n)_{ii}. \end{aligned} \tag{4}$$

Let us also define f_i as the probability that given $X_n = i$, there exists $k > 0$ such that the process will revisit i at time $n + k$:

$$f_i = P\{\exists k > 0 \text{ s.t. } X_{n+k} = i | X_n = i\}.$$

Note that f_i is independent of n because of time homegeniety.

Definition 12 *A state i is said to be* **recurrent** *if $f_i = 1$.*

In words, this means that given that the chain visits state i once, say at time n, then with probability 1 it will be in state i again at some later time $n + k$ $(k > 0)$. (We may not be able to put a bound on this k however).

Definition 13 *A state i is said to be* **transient** *if it is not recurrent. Equivalently state i is transient if $f_i < 1$.*

In words, this means that given that the chain visits state i once, say at time n, it is probable (with probability $1 - f_i$) that it will never be in state i after time n.

Theorem 14 *The following are equivalent.*

1. *i is recurrent.*
2. *$f_i = 1$.*
3. *$E(V_i \mid X_0 = i) = \infty$.*

4.

$$\sum_{n=1}^{\infty} (P^n)_{ii} = \infty.$$

Proof We do not give a rigorous proof, but merely outline the reasoning. Note that *1* is equivalent to *2* by definition.

If $X_n = i$, then the process will be in state i with probability f_i at a later time ($> n$). Now let us say the next visit happens at time $n + K_1 (> n)$, which itself is a random variable. Because of the Markov property, having revisited i at time $n + K_1$, the fact that we visited state i earlier at earlier time $n < (K_1 + n)$ is irrelevant, and we will revisit state i again with the same probability f_i, and that next visit will be at some other random time $n + K_2 > n + K_1$. (Note: There is a subtle point that we have overlooked - namely that the times $n+K_1$ and $n+K_2$ are random variables and not some fixed times. Our definition of Markov property (1) involved conditioning on a fixed - i.e. deterministic time n - and not on random times such as $n + K_1$ or $n + K_2$ etc. But it can be shown by advanced methods that the reasoning still works!)

Thus we can regard revisits to state i as Bernoulli trials with probability $(1-f_i)$ for a success (here a success means "not coming back to state i ever!"). Thus the total number of revisits V_i is geometrically distributed with pmf

$$P\{V_i = m\} = f_i^m (1 - f_i).$$

(i.e. Come back exactly m times, and then never again). It follows that

$$E(V_i) = \frac{1}{1 - f_i} \quad \text{if } f_i < 1,$$
$$E(V_i) = \infty \quad \text{if } f_i = 1.$$

In fact more strongly when $f_i = 1$ the number of revisits to state i is infinity with probability 1, i.e $P\{V_i = \infty\} = 1$, and $P\{V_i = m\} = 0$ for any finite number m. Thus the expected number of revisits to state i is ∞. Thus *2* implies *3*. Conversely suppose *3* is true, but $f_i < 1$. Then above discussion implies that the expected number of revisits $E(V_i)$ is finite which will contradict *3*. Hence *3* implies that $f_i = 1$ (same as *2*). Equation (4) shows that *3* and *4* are equivalent. ∎

The following corollary is obtained by negating every statement in the preceding theorem.

Corollary 15 *The following are equivalent.*

1. *i is transient.*
2. *$f_i < 1$.*
3. *$\lim_{k\to\infty} E(N_{ii}(k)) < \infty$. (i.e. The limit is finite).*
4. $$\sum_{k=1}^{\infty} (P^k)_{ii} < \infty.$$

   i.e. The sum converges.

Corollary 16 *The probability that state i will be revisited infinitely often is either 0 or 1 depending on whether i is transient or recurrent. No other value is possible!*

Proof In the proof of Theorem 14 we saw that the number of revisits V_i is a geometric random variable with success probability $1 - f_i$. Thus if $f_i < 1$ then

$$\begin{aligned} P\{V_i \text{ is finite}\} &= \lim_{k\to\infty} \sum_{m=1}^{k} f_i^{m-1}(1 - f_i) \\ &= (1 - f_i) \lim_{k\to\infty} \sum_{m=1}^{k} f_i^{m-1} = (1 - f_i)\frac{1}{1 - f_i} = 1. \end{aligned}$$

Thus $P\{V_i = \infty\} = 0$. If $f_i = 1$, then "success" never happens, so the only possible value for V_i is ∞. Thus $P\{V_i = \infty\} = 1$. ∎

Lemma 17 **recurrence** *and* **transience** *are class properties.*

See text book Corollary 4.2 for a proof.

Lemma 18 *If j is reachable from i, but i is not reachable from j, then i is transient.*

Proof Since $i \to j$ it follows that for some m, $(P^m)_{ij} > 0$. Further more if state j is visited after i had been visited, then no further visits to i can happen (since i is not reachable from j). Thus starting in i if j is visited after m times steps, it follows that the number of revisits V_i to i is less than m. Hence it follows that $P\{V_i < m\} \geq (P^m)_{ij} > 0$. Hence $P\{V_i = \infty\} \leq 1 - P\{V_i < m\} \leq 1 - (P^m)_{ij} < 1$. Then by Corollary 16, $P\{V_i = \infty\} = 0$, implying that state i is transient. ∎

Lemma 19 *A class that is not closed is transient.*

Proof If class C is not closed, then there exists a state $i \in C$ such that $i \to j \in C_2$ where $C_2 \neq C$. It follows that i is not reachable from j, other wise $i \leftrightarrow j$ and it would imply $C = C_2$ contradicting our assumption. Hence by Lemma 18, i is transient. Since transience is a class property, C is transient. ∎

Lemma 20 *A finite closed class is recurrent.*

Proof Suppose a finite closed class C is transient. The number of visits V_i to state $i \in C$ is finite with probability 1. Since C has only finite number of elements the total amount of time spent in class C is given by

$$\sum_{i \in C} V_i$$

is also finite with probability 1. This implies that the process must leave the class after a finite time. But this is not possible by the assumption of closedness. Hence (proof by contradiction) it follows that C must be recurrent. ∎

5 Periodicity and ergodicity

Definition 21 *Given a state i of a Markov chain, the* **period** $d(i)$ *of the state i is the greatest common divisor of the set*

$$\{n : (P^n)_{ii} > 0\}.$$

Note that $\{n : (P^n)_{ii} > 0\}$ is the set of all number of transitions after which it is possible to revisit i starting from i. As an example $d(i) = 2$ implies that it is not feasible to revisit state i starting from state i in odd number of transitions.

Definition 22 *A state i is said to be* **aperiodic** *if $d(i) = 1$ and* **periodic** *if $d(i) > 1$.*

Lemma 23 *Period is a class property. i.e. All states in a class have the same period.*

Lemma 24 *An irreducible chain that is aperiodic and positive recurrent is said to be* **ergodic**.

Definition 25 *A set of numbers $\pi_j \geq 0$ where $j(\in \mathcal{S})$ are the states are said to be* **stationary probabilities** *or (collectively) a* **stationary probability mass function** *for a Markov chain if they satisfy the* **stationarity equations**

$$\begin{aligned} \sum_j \pi_j &= 1, \\ \sum_i \pi_i P_{ij} &= \pi_j \quad \textit{for all } j. \end{aligned} \tag{5}$$

Note that the first equation is the condition for π_j to be a probability mass function, and that the second equation defines stationarity. If a stationary pmf exists, then if the chain is started with these π_j as the initial probabilities, i.e.

$$P\{X_0 = j\} = \pi_j$$

then for all time n, $P\{X_n = j\} = \pi_j$. We can also write the stationarity condition as

$$\pi P = \pi$$

as a row-vector matrix multiplication. Note that the vector π is a left eigenvector for the matrix P.

Lemma 26 *For an irreducible ergodic chain, a unique stationary probability mass function π_j exists. Also*

$$\lim_{n\to\infty} P\{X_n = j\} = \pi_j. \tag{6}$$

Consider the random variable $N_i(n)$ which is defined to be the number of visits to state i during the the times $0, 1, \ldots, n-1, n$. Then

$$\frac{\sum_{k=0}^{n} N_i(k)}{n}$$

is the **proportion of time spent** in state i up to time n and is a random variable. Note that by defining the indicator functions

$$\begin{aligned} I_i(k) &= 1 \quad \text{if } X_k = i, \\ &= 0 \quad \text{else,} \end{aligned}$$

and observing that $N_i(n) = \sum_{k=0}^{n} I_i(k)$, it follows that

$$E(N_i(k)) = \sum_{k=0}^{n} P\{X_k = i\}. \tag{7}$$

The **long run proportion of time spent on state** i is given by

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} N_i(k)}{n}, \tag{8}$$

if this limit exists. Note that in general the limit is a random variable.

Lemma 27 *For irreducible ergodic and for irreducible positive recurrent periodic chains the long run proportions given by* (8) *exist and are detrministic quantities given by the stationary probabilities* π_i. *In other words, with probability* 1

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} N_i(k)}{n} = \pi_i. \tag{9}$$

We are often interested in the long term behaviour of processes that depend on the state of a Markov chain. Mathematically we may be interested in the long term behaviour of $\{Y_n, n \geq 0\}$, a stochastic process whose state Y_n at time n depends on the state X_n (at the same time n) of Markov chain $\{X_n, n \geq 0\}$. This dependence may take different forms, and we shall consider a couple here.

Lemma 28 *Suppose* $\{X_n, n \geq 0\}$ *is an irreducible and ergodic Markov chain with limiting probabilities* π_j. *Let the stochastic process* $\{Y_n, n \geq 0\}$ *be defined by* $Y_n = f(X_n)$, *where* $f : \mathcal{S} \to \mathcal{S}'$ *is a bounded function. Here* $\mathcal{S}$ *is the state* *space of* X_n *and* $\mathcal{S}'$ *is the state space of* Y_n*, both being discrete. Suppose* $k \in \mathcal{S}'$ *is a possible value for* Y_n *and let* $f^{-1}\{k\} = \{j_1, j_2, \dots\} \subset \mathcal{S}$ *be the pre-image of* $\{k\}$ *under* f*. Then*

$$\lim_{n\to\infty} P\{Y_n = k\} = \sum_{j\in f^{-1}\{k\}} \pi_j.$$

Proof Since

$$P\{Y_n = k\} = \sum_{j\in f^{-1}\{k\}} P\{X_n = j\},$$

and since $\lim_{n\to\infty} P\{X_n = j\} = \pi_j$ exist, interchanging summation and limits we obtain the result. The interchanging is valid since the probabilities $P\{X_n = j\}$ are bounded for all n and j. ∎

Lemma 29 *Suppose* $\{X_n, n \geq 0\}$ *is an irreducible and ergodic Markov chain with (discrete) state space* $\mathcal{S}$ *and limiting probabilities* π_j*. Let the stochastic process* $\{Y_n, n \geq 0\}$ *with (discrete) state space* $\mathcal{S}'$ *be related to* $\{X_n, n \geq 0\}$ *in the following way.*

$$P\{Y_n = k | X_n = j\} = Q_{jk},$$

where $k \in \mathcal{S}'$ *and* $j \in \mathcal{S}$*. Then*

$$\lim_{n\to\infty} P\{Y_n = k\} = \sum_{j\in\mathcal{S}} Q_{jk}\pi_j.$$

Proof Clearly

$$P\{Y_n = k\} = \sum_{j\in\mathcal{S}} Q_{jk} P\{X_n = j\},$$

and taking limits and interchanging limits with summation we obtain the result. Interchanging is valid because Q_{jk} are bounded for all j and k (since they are conditional probabilities) and $P\{X_n = j\}$ are bounded for all n and j. ∎