HOME WORK 1 : MATH 490, Fall 2008

Due in class on Sep 24, Wed

Show your reasoning clearly in logical order.

Q1 For the following vectorfields find the largest domain such that each point has a neighborhood on which a flow map is defined for a common time interval.

(a)

$$f(x) = \sqrt{|x|}, \quad x \in \mathbb{R}.$$

(b)

$$f(x) = \frac{x}{|x|}, \quad x \in \mathbb{R}^n \setminus \{0\},$$
$$f(x) = 0, \quad x = 0.$$

(c) $f(x) = (\sqrt{|x_2|}, x_1)$, $x = (x_1, x_2) \in \mathbb{R}^2$.

(d)

$$f(x) = \frac{\sin(x)}{x} \quad x \in \mathbb{R} \setminus \{0\},$$
$$f(x) = 1, \quad x = 0.$$

Q2 Using the neighborhood existence theorem find a common time interval $[-a, a] \subset \mathbb{R}$ of existence of solutions to all initial conditions in the given compact (closed and bounded) regions for the following vectorfields. (Initial time assumed to be 0.)

(a) Vectorfield: $f(x) = \sqrt{|x|}$, $x \in \mathbb{R}$. Region $[\frac{3}{4}, \frac{5}{4}]$. Also find the *largest* common interval of existence and compare with the one guaranteed by the Theorem.

(b) Vectorfield: $f(x) = (\sqrt{|x_2|}, x_1^2)$, $x \in \mathbb{R}^2$. Region: Closed ball centered at $(0, 4)$ with radius 1.

Q3 Find the stable, unstable and center subspaces (if any) of the following vectorfields $\dot{x} = Ax$ for given A.

(a)

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

(c)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & -1 & -3 \end{bmatrix}.$$

Q4 You are given a vectorfield $f : \mathbb{R}^n \to \mathbb{R}^n$ that is locally Lipschitz. This means for each $x_0 \in \mathbb{R}^n$ there exists a $b > 0$ such that f is Lipschitz on the closed ball $B_b(x_0)$.

Let $x_0 \in \mathbb{R}^n$ and let $b > 0$ as above. Let $\phi(t, y)$ be the flow map defined on $[-a, a] \times B_b(x_0)$ where $a > 0$ as guaranteed by the *neighborhood existence theorem*.

Show that $\psi(t, y)$ defined on $[-a, a] \times B_b(x_0)$ by

$$\psi(t, y) = \phi(-t, y),$$

is the flow of the vectorfield $g = -f$. HINT: Use the integral equation form

$$\phi(t, y) = y + \int_0^t f(\phi(s, y))ds$$

and appropriate change of variables in the integral.

Q5 Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz (on $\mathbb{R}^n$) with constant K. Let the flow map $\phi(t, y)$ be defined on $\mathbb{R} \times B_b(x_0)$. Shown that for all $t \geq 0$ and $x, y \in \mathbb{R}^n$

$$e^{-Kt}|x - y| \leq |\phi(t, x) - \phi(t, y)| \leq e^{Kt}|x - y|.$$