SAMPLE FINALS QUESTIONS : MATH 301/0101, Fall 2008

You may assume basic trigonometric properties of functions sine and cosine as well as their continuity and differentiability on $\mathbb{R}$. Also you may assume that for all $x \in \mathbb{R}$

$$|\sin(x)| \leq |x|,$$

and that

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Q1 Use appropriate theorems on limit and continuity to prove that the following function has a limit at 1 and find the limit.

$$f(x) = \sin\left(\frac{\pi x^2 - \pi}{x-1}\right)$$

where f is defined for all nonzero real numbers x. You may assume that $\sin(x)$ is continuous on $\mathbb{R}$. State all theorems used.

Q2 Provide a direct $\epsilon - K$ proof that the sequence $\left(\sqrt{\frac{1}{n^2+1}}\right)$ is Cauchy. You may use the fact that

$$a > b > 0 \Leftrightarrow \sqrt{a} > \sqrt{b}.$$

Q3 For this problem you may assume that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$ and that $\sin(x)$ is continuous on $\mathbb{R}$.

(a) Use appropriate theorems (state these theorems and show the steps) to find the limit of the sequence

$$\left(\frac{\pi n - \sin(n^2)}{2n+1}\right).$$

You may assume that $\lim(1/n) = 0$.

(b) Use the above result and an appropriate theorem to find the limit of the following sequence:

$$\left(\sin\left(\frac{\pi n - \sin(n^2)}{2n+1}\right)\right).$$

Q4 You are given that a subset A of reals has supremum u. Prove that there exists a sequence (x_n) in A such that $\lim(x_n) = u$.

Q5 You are given the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^3 + x.$$

Provide a direct $\epsilon - \delta$ proof that f is continuous at $c \in \mathbb{R}$.

You are given a subset A of reals that is bounded. Is f uniformly continuous on A? Prove your answer.

Is f is uniformly continuous on $\mathbb{R}$? Prove your answer.

Q6 You are given the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 1,$$

for all rational numbers x, and

$$f(x) = -1,$$

for all irrational numbers x.

(a) Use the *sequential criterion* to prove that f does not have a limit at any $c \in \mathbb{R}$.

(b) Provide an $\epsilon - \delta$ proof that the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = x^2 f(x),$$

is differentiable at 0.

Q7 You are given the sequence $(x_n) = (-1 + 1/n^2)$. Show that (x_n) has a tail sequence that takes all its values in the set $(-1, -1/2)$.

Q8 You are given the set A defined by

$$A = \left\{ m - \frac{1}{n\pi} \mid m \in \mathbb{Z},\, n \in \mathbb{N} \right\}.$$

(a) What are the cluster points of A? (You need not prove, just state with brief explanation.).

(b) Does A contain any of its cluster points? (Explain your answer briefly. You may assume π is irrational).

(c) The function $f : A \to \mathbb{R}$ is defined by

$$f(x) = \sin(1/x).$$

Does f have a limit at 0?

Q9 You are given that $f : \mathbb{R} \to \mathbb{R}$ is bounded on $\mathbb{R}$. The $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$g(x) = x\, f(x).$$

Prove that g is continuous at 0.

Q10 You are given the recursivley defined sequence $x_1 > 0$

$$x_{n+1} = \sqrt{x_n}, \ \forall n \in \mathbb{N}.$$

Show that (x_n) converges and find the limit.

(HINT: You may use $a > b > 0 \Rightarrow \sqrt{a} > \sqrt{b}$. Investigate cases $x_1 > 1, x_1 = 1$ and $x_1 < 1$.)

Explain your steps and state all theorems used.

Q11 Prove that the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = 1/x$ is unbounded on $(0, \infty)$.

Q12 You are given $f : A \to \mathbb{R}$, where A is given by

$$A = \{(-1)^n/n |\ n \in \mathbb{N}\},$$

and

$$f(x) = \sin(\pi/x).$$

Does f have a limit at 0? (Note that 0 is a cluster point of A). Prove your answer.

Q13 You are given the sequence (x_n) defined by

$$x_{n+1} = \frac{1}{2}\sin(x_n)$$

where $x_1 \in \mathbb{R}$. Prove that (x_n) is contractive. Explain why it is convergent. Find the limit. Show all the steps and state all the theorems used in finding the limit.