Problem 1: Improved Euler’s Method (5 POINTS)  Show all Work!

\[ \frac{dy}{dx} = y \quad y(0) = 1 \]

a) Use the improved Euler’s Method to approximate the solution of this IVP (namely \( e^x \)) at the point \( x=1 \). Assume \( h=0.25 \)

\[
y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_n, y_n + hf(x_n, y_n)))
\]

\[
y_{n+1} = y_n + \frac{h}{2}(y_n + y_n + hy_n)
\]

\[
y_{n+1} = y_n(1 + h + \frac{h^2}{2})
\]

\[
y_4 = (\frac{40}{32} + \frac{1}{32})^4 = (\frac{41}{32})^4
\]

b) \( e^1 = 2.718281828 \). So what is \( \frac{\text{error}}{h} \)?

\[
[2.718281828 - (\frac{41}{32})^4] \times 4
\]

c) What order is the improved Euler’s method? and hence what do you expect to happen to the ratio \( \frac{\text{error}}{h} \) as \( h \to 0 \)?

Second order. \( \text{error} = ch^2 \) so \( \frac{\text{error}}{h} = \frac{ch^2}{h} = ch \to 0 \) as \( h \to 0 \)
Problem 1: Euler’s Method Workspace
Problem 2: Auxiliary equations and Implication of Roots (15 POINTS) Show all Work!

\[ y'' + \alpha y' + 4y = e^{-t} \quad y(0) = 1 \text{ and } y'(0) = 0 \]

a) Write down the auxiliary equation for the corresponding homogeneous system of the IVP above.

\[ r^2 + \alpha r + 4 = 0 \]

b) Determine the values of \( \alpha \) so the auxiliary equation has:

\[ \frac{-\alpha \pm \sqrt{\alpha^2 - 16}}{2} = r_{1,2} \]

Case 1: A double real root

\[ \alpha = \pm 4 \Rightarrow r = \frac{-\alpha}{2} \Rightarrow r = \mp 2 \]

Case 2: Two real roots

\[ \alpha^2 - 16 > 0 \Rightarrow \alpha^2 > 16 \Rightarrow |\alpha| > 4 \]

Case 3: Complex conjugate pair roots

\[ \alpha^2 - 16 < 0 \Rightarrow \alpha^2 < 16 \Rightarrow |\alpha| < 4 \]

c) What are the roots for each case above.

Please leave \( \alpha \) as a parameter in your answers but describe its constraint in each case.

Case 1: \( r = 2 \) for \( \alpha = -4 \), \( r = -2 \) for \( \alpha = 4 \)

Case 2: \( r_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 16}}{2} \) for \( |\alpha| > 4 \)

Case 3: \( r_{1,2} = \frac{-\alpha \pm i\sqrt{16 - \alpha^2}}{2} \) for \( |\alpha| < 4 \)

d) Write down the homogeneous solution for each case above.

Case 1: \( y_h = c_1 e^{-2t} + C_2 te^{-2t} \) and \( y_h = c_1 e^{2t} + C_2 e^{2t} \)

Case 2: \( c_1 e^{\frac{-\alpha + \sqrt{\alpha^2 - 16}}{2}} + c_2 e^{\frac{-\alpha - \sqrt{\alpha^2 - 16}}{2}} \)

Case 3: \( e^{\frac{-\alpha}{2}t}(c_1 \sin(\frac{\sqrt{16 - \alpha^2}}{2} t) + c_2 \cos(\frac{\sqrt{16 - \alpha^2}}{2} t) \)
Problem 2: Auxiliary equations and Implication of Roots Workspace
Problem 3: Method of Undetermined Coefficients (15 POINTS) Show all Work!

\[ y'' + \alpha y' + 4y = e^{-t} \quad y(0) = 1, \; y'(0)=0 \]

a) Determine the particular solution to the ODE above if 
\( \alpha = 0 \): plug guess into ODE

\[ y_p = Ae^{-t} \implies A + 4Ae^{-t} = e^{-t} \implies A = \frac{1}{5} \implies y_p = \frac{1}{5}e^{-t} \]

\( \alpha = 4 \): Note \( e^{-t} \) is not a solutions to the homogenous equations for this \( \alpha \) so \( y_p = Ae^{-t} \) is still ok.

\[ 4y = 4Ae^{-t} \]
\[ 4y' = -4Ae^{-t} \]
\[ y'' = Ae^{-t} \]

plugging into ODE \( \implies Ae^{-t} = e^{-t} \implies A = 1 \)
\[ y_p = e^{-t} \]

\( \alpha = 6 \):
\[ 4y = 4Ae^{-t} \]
\[ 6y' = -6Ae^{-t} \]
\[ y'' = Ae^{-t} \]

plugging into ODE \( \implies -Ae^{-t} = e^{-t} \implies A = -1 \)
\[ y_p = -e^{-t} \]

using the Method of Undetermined Coefficients.

b) What is the general solution to the ODE for \( \alpha = 4 \).

\[ y_g = y_h + y_p = c_1 e^{-2t} + c_2 te^{-2t} + e^{-t} \]

c) Determine the solution to the IVP for \( \alpha = 4 \). (apply initial conditions)

\[ y_g(0) = 1 = c_1 + 1 \implies c_1 = 0 \]
\[ y_g = c_2 te^{-2t} + e^{-t} \]
\[ y'_g = c_2(e^{-2t} - 2te^{-2t}) - e^{-t} \]
\[ y_g(0) = 0 = c_2(1 - 0) - 1 \implies c_2 = 1 \]
\[ y_g = te^{-2t} + e^{-t} \]
Problem 3: Method of Undetermined Coefficients Workspace
Problem 4: Mass-Spring Oscillator and Phase Plane (20 POINTS) Show all Work!

a) Draw a sketch of a Mass-Spring Oscillator with a single mass (1kg) and single spring (k=2) attaching the mass to a wall. Assign y=0 to the rest position of the mass.

b) Derive the second-order ODE equation of motion of the mass for the case when there is dampening called \( b = 3 \) and \textbf{NO} forcing function.

\[
\sum F = ma \Rightarrow F_{\text{spring}} - F_{\text{dampening}} = ma \Rightarrow -ky - by' = m \cdot y'' \Rightarrow y'' + 3y' + 2y = 0
\]

c) Using the assignments \( x_1 = y \) and \( x_2 = y' \) decompose this single second-order ODE into two first order ODEs.

\[
x_1' = y' = x_2 \\
x_2' = y'' = -2x_1 - 3x_2
\]

d) Find the critical points for this system.

\( x_1' = 0 \longrightarrow x_2 = 0 \) and then \( x_2' = 0 \longrightarrow x_1 = 0 \) so \((0, 0)\) is the only critical point.

e) Put the system into matrix normal form. (Keep you variables straight)

\[
\begin{pmatrix}
x_1' \\
x_2'
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-2 & -3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]
Problem 4: Mass-Spring Oscillator and Phase Plane Workspace

f) Solve the system using eigenvalues and eigenvectors for the vector equation homogeneous solution.

\[ |A - rI| = \begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix} - \begin{vmatrix} r & 0 \\ 0 & r \end{vmatrix} = \begin{vmatrix} -r & 1 \\ -2 & -(3 + r) \end{vmatrix} = 0 \]

\[ r^2 + 3r + 2 = 0 \]
\[ (r + 1)(r + 2) = 0 \]
\[ r_1 = -1, r_2 = -2 \]

\[ r_1 = -1 \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow u = s \begin{pmatrix} -1 \\ 1 \end{pmatrix} \]

\[ r_2 = -2 \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow u = s \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \]

\[ \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \]

\[ \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}c_2 \\ c_2 \end{pmatrix} \]

\[ -c_1 - \frac{1}{2}c_2 = 0 \rightarrow c_1 = -\frac{1}{2}c_2 \]
\[ c_1 + c_2 = 1 \rightarrow -\frac{1}{2}c_2 + c_2 = 1 \rightarrow c_2 = 2 \rightarrow c_1 = 1 \]

\[ y(t) = e^{-t} - e^{-2t} \]
\[ y'(t) = -e^{-t} + 2e^{-2t} \]
Problem 5: Variation of Parameters (10 POINTS)  Show all Work!

\[ y'' - y' - 2y = e^{3t} \]

Use variation of parameters to find the particular solution for this non-homogeneous ODE. Show all steps of the method.

\[ r^2 - r - 2 = 0 \]
\[ (r - 1)(r + 1) = 0 \quad \rightarrow \quad y_h(t) = c_1 e^{2t} + c_2 e^{-t} \]

\[ y_p(t) = v_1 y_1 + v_2 y_2 = v_1 e^{2t} + v_2 e^{-t} \text{ variational form of particular solution} \]

\[ v'_1 e^{2t} + v'_2 e^{-t} = 0 \quad (1) \text{ variational equations.} \]
\[ 2v'_1 e^{2t} - v'_2 e^{-t} = e^{3t} \quad (2) \]

\[ v'_1 = -v'_2 e^{3t} \text{ from equation (1) plugging into (2)} \]
\[ -2v'_2 e^{-t} - v'_2 e^{-t} = e^{3t} \quad \rightarrow \quad v'_2 = -\frac{e^{4t}}{3} \quad \rightarrow \quad v'_1 = \frac{e^t}{3} \]

\[ v_1 = \frac{e^t}{3} \quad v_2 = -\frac{e^{4t}}{12} \]

\[ y_p(t) = \frac{e^t}{3} e^{2t} - \frac{e^{4t}}{12} e^{-t} = \frac{e^{3t}}{4} \]
Problem 6: Solving IVP using Laplace transform (15 POINTS)
Show all Work!

Solve the IVP using Laplace transform methods.

\[ y'' - 2y' + 5y = -8e^{-t} \quad y(0) = 2 \text{ and } y'(0) = 12 \]

See Example (1) page 377 of text.
Problem 6: Workspace
Problem 7: Definition and properties of Laplace transform (20 POINTS)  Show all Work!

a): What is the definition of the Laplace transform?

See page 351 of text.

b): Using the definition, derive the Laplace transform of the function \( t \).

\[
\int_0^\infty te^{-st}dt = \lim_{N \to \infty} \left( \frac{-te^{-st}}{s} |_0^N + \frac{1}{s} \int_0^N e^{-st}dt \right) = \lim_{N \to \infty} \left( \frac{-te^{-st}}{s} |_0^N - \frac{e^{-st}}{s^2} |_0^N \right) = \frac{1}{s^2}
\]

c): Graph the following discontinuous function, then express it using unit step functions.

\[
f(t) = \begin{cases} 
5, & 0 < t < 1 \\
t - 1, & 1 < t < 2 \\
t^2, & t \geq 2 
\end{cases}
\]

\[
f(t) = 5u(t) - 5u(t - 1) + (t - 1)u(t - 1) - (t - 1)u(t - 2) + t^2u(t - 2)
\]

d): Using the properties of the unit step, what is the Laplace transform of \( f(t) \).

\[
f(t) = 5u(t) - 5u(t - 1) + (t - 1)u(t - 1) - (t - 1)u(t - 2) + t^2u(t - 2)
\]

\[
f(t) = 5u(t) - 5u(t - 1) + (t - 1)u(t - 1) + (t^2 - t + 1)u(t - 2)
\]

\[
f(t) = 5u(t) - 5u(t - 1) + (t - 1)u(t - 1) + [(t^2 - 4t + 4) + 3(t - 2) + 3]u(t - 2)
\]

\[
f(t) = 5u(t) - 5u(t - 1) + (t - 1)u(t - 1) + (t - 2)^2u(t - 2) + 3(t - 2)u(t - 2) + 3u(t - 2)
\]

\[
f(t) = \frac{5}{s} - \frac{5e^{-s}}{s} + \frac{e^{-s}}{s^2} + \frac{2e^{-2s}}{s^3} + \frac{3e^{-2s}}{s^2} + \frac{3e^{-2s}}{s}
\]
Problem 7: Worksheet
Problem 8: Solve using separation of variables. (10 POINTS) Show all Work!

Solve the following IVP using separation of variables.

\[ y' = x^3(1 - y) \quad y(0) = 3 \]

\[
\frac{dy}{dx} = x^3(1 - y) \\
\int \frac{dy}{1 - y} = \int x^3 \, dx \\
-\ln(1 - y) = \frac{x^4}{4} + c \\
\ln(1 - y) = -\frac{x^4}{4} + c \\
1 - y = ce^{-\frac{x^4}{4}} \\
y = 1 - ce^{-\frac{x^4}{4}} \\
y(0) = 1 - c = 3 \rightarrow c = -2 \quad y = 1 + 2e^{-\frac{x^4}{4}}\]
Problem 9: Solve using Integrating Factors (10 POINTS) Show all Work!

Solve the following IVP using Integrating factors

\[ \frac{dy}{dx} + 4y - e^{-x} = 0 \quad y(0) = \frac{4}{3} \]

\[ \mu = e^{\int 4dx} = e^{4x} \]

\[ e^{4x} \left( \frac{dy}{dx} + 4y \right) = e^{4x} e^{-x} \]

\[ e^{4x} \frac{dy}{dx} + 4e^{4x} y = \frac{d(ye^{4x})}{dx} = e^{3x} \]

\[ \int \frac{d(ye^{4x})}{dx} = \int e^{3x} \]

\[ ye^{4x} = \frac{e^{3x}}{3} + c \]

\[ y = \frac{e^{-x}}{3} + ce^{-4x} \]

\[ y(0) = \frac{1}{3} + c = \frac{4}{3} \quad \rightarrow \quad c = 1 \]

\[ y(x) = \frac{e^{-x}}{3} + e^{-4x} \]
Problem 10: Identify and Solve Exact Equations (10 POINTS)  
Show all Work!

Show the following problem is exact and then solve it.

\[(\frac{t}{y})dy + (1 + \ln(y))dt = 0\]

\[\frac{\partial}{\partial t} \left( \frac{t}{y} \right) = \frac{1}{y} = \frac{\partial(1+\ln(y))}{\partial y}\]

Exact

\[F(t, y) = \int \frac{t}{y}dy + f(t)\]

\[F(t, y) = t \ln(y) + f(t)\]

\[\frac{\partial F(t,y)}{\partial t} = \ln(y) + f'(t)\]

\[1 + \ln(y) = \ln(y) + f'(t) \rightarrow f'(t) = 1 \rightarrow f(t) = t\]

\[F(t, y) = t \ln(y) + t\]

solution is \[F(t, y) = C = \Rightarrow t \ln(y) + t = C\]
Problem 11: True or False and Short Answer (20 POINTS)  Show all Work!
If false, you must explain why to get any credit.

a) If the Wronskian of two solutions of an ODE is 0 at any point \( t \in I \) then the two solutions are linearly independent.

FALSE: two solutions to an ODE are independent iff the Wronskian is nonzero at every point in the interval.

b) Two vectors \( x \) and \( y \) are linearly independent if \( c_1 x + c_2 y = 0 \) implies \( c_1 = c_2 = 0 \)

TRUE

c) \( y'' + y^3 = 0 \) is a second order linear ODE.

FALSE: the \( y^3 \) term does not fit into the model for linear ODE. (ie. is not a linear term)

d) The existence and uniqueness theory for the IVP \( \frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0 \)

assumes \( f \) needs only to be continuous on a rectangle for the IVP to have a unique solution in some neighborhood of \((x_0, y_0)\).

FALSE: \( f \) also need to have the partial derivative \( \frac{\partial f}{\partial y} \) continuous.

e) Short Answer: Draw a block diagram representing the process of using the Laplace Transform in solving IVP problems. Explain the use of initial conditions in this process. Also explain when this process is not applicable in solving an IVP.

See the block diagram on page 351 of the book. The initial conditions are used in transforming the derivatives of the ODE. The process fails if the functions involved are NOT piecewise continuous exponential order.