A generalized $p$-value approach to inference on common mean

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Abstract

Statistical inference for the common mean of several univariate normal populations has many useful applications in practice. In this paper, we develop a generalized $p$-value approach to derive tests and confidence intervals for the common mean based on several well known estimates of the common mean. The case of a multivariate normal distribution with a common component mean is also addressed and a generalized $p$-value approach based on the maximum likelihood estimate of the common mean is developed. We also report results of a simulation study showing powers and coverage probabilities.

Keywords and phrases: Cholesky decomposition, common mean, confidence interval, generalized $p$-value, maximum likelihood, Wishart distribution.

AMS Subject Classification: Primary 62H15, Secondary 62H10

1 Introduction

In this paper we consider the problem of providing an exact test and a confidence interval of the common mean of several univariate normal populations on the basis of a generalized $p$-value approach. This problem has been extensively discussed in the literature [Graybill and Deal (1959), Khatri and Shah (1974), Norwood and Hinkelmann (1977), Bhattacharya...
(1980), Pal and Sinha (1996), Mitra and Sinha (2007)]. Some important aspects of estimation of the common mean from a decision theoretic point of view have also been addressed by several authors [Sinha and Mouqadem (1982), Sinha (1979, 1985), Zacks (1966, 1970), Pal and Lim (1997)]. Regarding exact confidence intervals for the common mean, Jordan and Krishnamoorthy (1996) discussed procedures based on linear combinations of \( t \) and \( F \) statistics, while Yu, Sun and Sinha (1999) considered this problem based on the combination of associated \( p \)-values. Some recent contributions from a generalized \( p \)-value approach are due to Krishnamoorthy and Lu (2003) and Lin and Lee (2005).

Suppose we have \( k \) independent normal populations with a common mean \( \mu \) and variances \( \sigma_i^2 \) for \( i = 1, 2, \ldots, k \). Let \((\bar{x}_i, s_i^2)\) be the sample mean and the sample variance of \( i \)th population based on a sample of size \( n_i \). Let \( D_i = \bar{x}_i - \bar{x}_1 \) for \( i = 2, \ldots, k \). Let \( C_u \) be the general class of unbiased estimates of \( \mu \), defined as \( C_u = \{ \hat{\mu}_\phi : \hat{\mu} = \bar{x}_1 + \sum_{i=2}^{k} D_i \phi_i(s_i^2, s_2^2, \ldots, s_k^2; D_2^2, \ldots, D_k^2) \} \). Also let \( C_0 \) be a subset of the general class of unbiased estimates of \( \mu \), defined as \( C_0 = \{ \hat{\mu}_\phi : \hat{\mu} = \sum_{i=1}^{k} \bar{x}_i \phi_i(s_1^2, s_2^2, \ldots, s_k^2) \} \), with \( \sum_{i=1}^{k} \phi_i = 1 \). Clearly \( C_0 \subset C_u \). The standard Graybill-Deal estimate of \( \mu \) is given by

\[
\hat{\mu}_{GD} = \left[ \sum_{i=1}^{k} \frac{n_i \bar{x}_i}{s_i^2} \right] / \left[ \sum_{i=1}^{k} \frac{n_i}{s_i^2} \right]
\]

which obviously belongs to \( C_0 \). Sinha and Mouqadem (1982) also considered several other unbiased estimates of \( \mu \). Four popular estimates of \( \mu \) in the context of two independent normal distributions with equal sample sizes are given below.

\[
\hat{\mu}_1 = \left[ \frac{\bar{x}_1}{s_1^2} + \frac{\bar{x}_2}{s_2^2} \right] / \left[ \frac{1}{s_1^2} + \frac{1}{s_2^2} \right] \quad \text{[Graybill – Deal, 1959]}
\]

\[
\hat{\mu}_2 = \bar{x}_1 + D \frac{s_1^2 + D^2}{s_1^2 + s_2^2 + D^2} \quad \text{[Sinha – Mouqadem, 1982]}
\]

\[
\hat{\mu}_3 = \bar{x}_1 + D_{\text{min}}(0.5, \frac{s_1^2}{s_1^2 + s_2^2}) \quad \text{[Sinha, 1979]}
\]

\[
\hat{\mu}_4 = \bar{x}_1 + D \frac{s_1}{s_1 + s_2} \quad \text{[Sinha – Mouqadem, 1982]}
\]

It should be noted that \( \hat{\mu}_1, \hat{\mu}_3, \hat{\mu}_4 \) belong to \( C_0 \) and \( \hat{\mu}_2 \) belongs to \( C_u \), but not to \( C_0 \).
Our main objective in this paper is to use the generalized $p$-value approach based on a very general form of the unbiased estimate of the common mean given above in both $C_0$ and $C_u$ to construct exact tests and confidence intervals for the common mean. This includes some results of Lin and Lee (2005) as a special case. As applications, we mention below two examples of data sets dealing with such a common mean problem.

**Example 1.** Our first example is due to Meier (1953). The results are the outcomes of four experiments used to estimate the mean percentage of albumin in the plasma protein of normal human subjects. The assumption is that the distributions are normal with a common mean $\mu$. The problem is to infer about $\mu$ based on available data.

**Example 2.** Our second example [Eberhardt et al. (1989)] deals with the problem of estimation of mean selenium in non-fat milk powder by combining results of four methods: atomic absorption spectrometry, neutron activation by instrumental and radiochemical, and isotope dilution mass spectrometry.

Our second objective in this paper is to consider a multivariate normal population with a common component mean and an unknown dispersion matrix, and provide exact tests and confidence intervals for the common mean based on the generalized $p$-value approach. This is based on the well known Cholesky decomposition of a Wishart matrix. An application of this set up appeared in an EPA study [Yu, Sun and Sinha (2004)] which dealt with inference about mean reid vapor pressure (RVP) based on correlated RVP measurements collected from fields and laboratories.

The organization of the paper is as follows. For the sake of completeness, we provide a brief account of the generalized $p$-value procedure in Section 2. Sections 3 and 4 contain main results under the two classes $C_0$ and $C_u$, respectively. In Section 5 we consider the case of the multivariate normal population. Some simulation results comparing several competing procedures for two univariate normal populations are provided in Section 6. This section also contains the simulation results providing the performance of a proposed...
procedure for a bivariate normal population with a common mean. Finally, we discuss an application in Section 7.

2 Generalized \( p \)-value and generalized confidence intervals

The concept of the Generalized \( p \)-value was first introduced by Tsui and Weerahandi (1989) to deal with some nontrivial statistical testing problems. These problems involve nuisance parameters in such a fashion that the derivation of a standard pivot is not possible.

Let \( X \) be the random variable with density function, \( f(X|\zeta) \) where \( \zeta = (\theta, \eta) \), \( \theta \) is our parameter of interest and \( \eta \) is possibly a vector of nuisance parameters.

Suppose we have the following hypothesis to test:

\[
H_0 : \theta \leq \theta_0 \quad vs. \quad H_1 : \theta > \theta_0,
\]

where \( \theta_0 \) is specified value.

Let \( x \) be the observed value of the random variable \( X \). \( T = T(X; x, \zeta) \) is said to be a generalized test statistic if following three properties hold.

1. For fixed \( x \) and \( \zeta = (\theta_0, \eta) \), the distribution of \( T(X; x, \zeta) \) is independent of nuisance parameter \( \eta \).
2. \( t_{obs} = T(x; x, \zeta) \) does not depend on unknown parameters.
3. For fixed \( x \) and \( \eta \), \( P[T(X; x, \zeta) \geq t] \) is either stochastically increasing or decreasing in \( \theta \) for any given \( t \).

If \( T \) is stochastically increasing in \( \theta \), the generalized \( p \)-value is defined as

\[
p = \sup_{\theta \leq \theta_0} P[T(X; x, \theta, \eta) \geq t] = P[T(X; x, \theta_0, \eta) \geq t]
\]

(2)

where \( t = T(x; x, \theta_0, \eta) \).

To derive a confidence interval for \( \theta \), let \( T_c(X; x, \theta, \eta) \) satisfy the following conditions:

1. The distribution of \( T_c(X; x, \theta, \eta) \) does not depend on any unknown parameters.
2. The observed value of \( T_c(X; x, \theta, \eta) \) is free of the nuisance parameter.
Then, we say $T_c(X; x, \theta, \eta)$ is a generalized pivotal quantity, and if $t_1$ and $t_2$ are such that

$$P[t_1 \leq T_c(X; x, \theta, \eta) \leq t_2] = 1 - \alpha$$

then $\Theta = \{\theta : t_1 \leq T_c(X; x, \theta, \eta) \leq t_2\}$ is a generalized confidence interval with confidence coefficient $1 - \alpha$. For example, if $T_c(x; x, \theta, \eta) = \theta$, then $[T_c(x; \alpha/2), T_c(x; 1 - \alpha/2)]$ is $1 - \alpha$ confidence interval for $\theta$, where $T_c(x; \alpha/2)$ stands for $\alpha/2^{th}$ quantile.

### 3 Common mean problem in $C_0$

In this section, we provide exact tests and confidence intervals for the common mean of $k$ univariate normals based on unbiased estimates in $C_0 = \{\hat{\mu}_0 : \hat{\mu} = \sum_{i=1}^k \bar{x}_i \phi_i(s_i^2, s_{i+1}^2, \ldots, s_k^2)\}$, with $\sum_{i=1}^k \phi_i = 1$. Note that $\bar{X}_i = \sum_{j=1}^{n_i} X_{ij} / n_i \sim N(\mu, \sigma_i^2 / n_i)$ and $V_i / \sigma_i^2 \sim \chi_{n_i-1}^2, i = 1, 2, \ldots, k$ where $V_i = \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2 = (n_i - 1)S_i^2$.

Clearly the $V_i$'s and $\bar{X}_i$'s are independent. Let $V = (V_1, V_2, \ldots, V_k)$, and $v = (v_1, \ldots, v_k)$ be the observed value of $V$. Let $\bar{X} = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k)$, and $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_k)$ be the observed value of $\bar{X}$.

For a given $\phi(.)$, let $\hat{\mu}^* = \sum_{i=1}^k \bar{x}_i \phi_i(s_1^2, s_2^2, \ldots, s_k^2)$ so that the distribution of $\hat{\mu}^*$ is $N\left(\mu, \sum_{i=1}^k n_i \phi_i^2(s_1^2, s_2^2, \ldots, s_k^2)\right)$.

Let $I = I(\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2) = \sum_{i=1}^k \sigma_i^2 n_i \phi_i^2(s_1^2, s_2^2, \ldots, s_k^2)$. Then $Z = \frac{(\hat{\mu}^* - \mu)}{\sqrt{\sigma^2}} \sim N(0, 1)$. Finally, we define $I^*$ as

$$I^*(V; v; \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2) = \sum_{i=1}^k \frac{\sigma_i^2 V_i}{n_i} \phi_i^2(s_1^2 V_1, s_2^2 V_2, \ldots, s_k^2 V_k).$$

We are now ready to define a generalized pivotal quantity for confidence intervals of the common mean $\mu$ through $\hat{\mu}^*$ and $I^*$ as
\[ T_c(\bar{X}, V; \bar{x}, v, \mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2) = \sum_{i=1}^{k} \bar{x}_i \phi_i(\sigma_1^2 v_1, \sigma_2^2 v_2, \ldots, \sigma_k^2 v_k) - \frac{\sqrt{T}(\hat{\mu} - \mu)}{\sqrt{T}} \]

\[ = \sum_{i=1}^{k} \bar{x}_i \phi_i(\sigma_1^2 v_1, \sigma_2^2 v_2, \ldots, \sigma_k^2 v_k) \]

\[ - Z \left( \sum_{i=1}^{k} \frac{v_i}{n_i} \phi_i^2(\sigma_1^2 v_1, \sigma_2^2 v_2, \ldots, \sigma_k^2 v_k) \right). \]  

(4)

It can be easily shown that the two properties of generalized pivotal quantity as mentioned in Section 2 are satisfied because the distribution of \( T_c(\cdot) \) does not depend on any unknown parameters and the observed value of \( T_c(\bar{x}, v; \bar{x}, v) = \mu \). So for given \( \bar{x} \) and \( v \), \( T_c \) is a generalized pivotal quantity. Hence, using (3), we can find \( t_1 \) and \( t_2 \) such that \( \Theta = \{ \theta : t_1 \leq T_c(X; x, \theta, \eta) \leq t_2 \} \) is a generalized confidence interval for \( \mu \) with confidence coefficient \( 1 - \alpha \). In particular, we can choose \( t_1 = T_c(x; \alpha/2) \) and \( t_2 = T_c(x; 1 - \alpha/2) \) where \( T_c(x; \alpha/2) \) stands for \( \alpha/2^{th} \) quantile of the pivot \( T_c \).

Consider the following hypotheses:

\[ H_0 : \mu \leq \mu_0 \quad \text{vs.} \quad H_1 : \mu > \mu_0. \]

From \( T_c \), we can derive a generalized test variable, \( T \), as \( T = T_c - \mu \). So the first two properties are automatically satisfied. Since the distributions of the random variables \( V_1, \ldots, V_k, Z \) are free of parameters, it can be shown that \( P[T \leq t; \mu| \bar{x}, v] = P[T \leq \mu| \bar{x}, v] \) which is an increasing function of \( \mu \). Hence the generalized \( p \)-value can be obtained as

\[ p = P[T \leq t| \mu = \mu_0; \bar{x}, v] = P[T \leq \mu_0| \bar{x}, v]. \]  

(5)

\( H_0 \) is rejected if \( p < \alpha \).

For the two sided alternative, the generalized \( p \) value can be defined as

\[ p = 2 * \min\{P[T < \mu_0| \bar{x}, v], P[T > \mu_0| \bar{x}, v]\}. \]  

(6)
4 Common mean problem in \( C_u \)

In this section, we provide exact tests and confidence intervals for the common mean of \( k \) univariate normals based on unbiased estimates in the general class \( C_u \). Recall that 
\[
C_u = \{ \mu_\phi : \mu = \bar{x}_1 + \sum_{i=2}^{k} D_i \phi_i (s_1^2, s_2^2, \ldots, s_k^2; D_2^2, \ldots, D_k^2) \}.
\]
Moreover, \( \mu_2 \) belongs to \( C_u \). Our approach is very similar to the one in Section 3.

Let 
\[
\hat{\mu}^* = \sum_{i=1}^{k} \bar{X}_i \phi_i \left( \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2; \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}, \ldots, \frac{\sigma_k^2}{n_k} \right).
\]
Obviously, 
\[
\hat{\mu}^* \sim N \left( \mu, \sum_{i=1}^{k} \frac{\sigma_i^2}{n_i} \phi_i^2 \left( \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2; \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}, \ldots, \frac{\sigma_k^2}{n_k} \right) \right).
\]

Let 
\[
I = I \left( \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2; \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}, \ldots, \frac{\sigma_k^2}{n_k} \right)
\]
\[
= \sum_{i=1}^{k} \frac{\sigma_i^2}{n_i} \phi_i^2 \left( \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2; \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}, \ldots, \frac{\sigma_k^2}{n_k} \right).
\]

Then clearly 
\[
Z = \frac{(\hat{\mu}^* - \mu)}{\sqrt{I}} \sim N(0, 1).
\]

We now define \( I^* \) as 
\[
I^* = I^* \left( \mathbf{V}, \mathbf{v}; \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2 \right)
\]
\[
= \sum_{i=1}^{k} \frac{\sigma_i^2 v_i}{n_i V_i} \phi_i^2 \left( \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2; \frac{\sigma_1^2 v_1}{n_1 V_1} + \frac{\sigma_2^2 v_2}{n_2 V_2}, \ldots, \frac{\sigma_k^2 v_k}{n_k V_k} \right).
\]

Now using a similar technique as in the preceding section, we can define a generalized pivotal quantity for confidence interval of the common mean \( \mu \) as 
\[
T_c (\bar{X}, \mathbf{V}; \bar{\mathbf{x}}, \mathbf{v}, \mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2)
\]
\[
= \sum_{i=1}^{k} \bar{x}_i \phi_i \left( \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2; \frac{\sigma_1^2 v_1}{n_1 V_1} + \frac{\sigma_2^2 v_2}{n_2 V_2}, \ldots, \frac{\sigma_k^2 v_k}{n_k V_k} \right) - \sqrt{I^*} (\hat{\mu}^* - \mu)
\]
\[
= \sum_{i=1}^{k} \bar{x}_i \phi_i \left( \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2; \frac{\sigma_1^2 v_1}{n_1 V_1} + \frac{\sigma_2^2 v_2}{n_2 V_2}, \ldots, \frac{\sigma_k^2 v_k}{n_k V_k} \right) - \sqrt{Z} \left( \sum_{i=1}^{k} \frac{\sigma_i^2 v_i}{n_i V_i} \phi_i^2 \left( \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2; \frac{\sigma_1^2 v_1}{n_1 V_1} + \frac{\sigma_2^2 v_2}{n_2 V_2}, \ldots, \frac{\sigma_k^2 v_k}{n_k V_k} \right) \right). \tag{7}
\]

It should be noted that the distribution of \( T_c \) does not depend on any unknown parameters and the observed value, i.e., \( T_c (\bar{\mathbf{x}}, \mathbf{v}; \bar{\mathbf{x}}, \mathbf{v}) \) is equal to \( \mu \). So \( T_c \) is a generalized
pivotal quantity. Hence we can find the generalized $p$-value and power for hypothesis testing problem for $\mu$ as well as a generalized confidence interval for $\mu$ by following the procedure described in the preceding section.

5 Multivariate normal set up

Suppose we have $n$ observations, $(X_1, X_2, \ldots, X_n)$, from a $k$ variate normal distribution given by $N(\mu\mathbf{1}, \Sigma)$ where $\mu$ is the common mean. Let $S = \sum_{i=1}^{n} (X_i - \bar{X})(X_i - \bar{X})'$ be the matrix of sample sum of squares and products, which follows a Wishart distribution with parameters $m = n - 1$ and $\Sigma$.

If $\Sigma$ is known, the maximum likelihood estimate (MLE) of $\mu$ is given by $\hat{\mu}^* = \frac{\Sigma^{-1}\bar{X}}{\Sigma^{-1} + n}$. When $\Sigma$ is unknown, we can replace it by $S/(n - 1)$ which gives $\hat{\mu} = \frac{S^{-1}\bar{X}}{S^{-1} + 1}$. It is shown in Halperin (1961) that $\hat{\mu}$ is MLE.

Although $\hat{\mu}$ is unbiased for $\mu$, it is obvious that the variance of $\hat{\mu}$ and its exact distribution are quite complicated, and hence it is not obvious how one should proceed to provide exact inference for $\mu$ based on $\hat{\mu}$. Exact tests and confidence intervals can be constructed using some properties of the underlying multivariate normal distributions [Halperin (1961), Krishnamoorthy and Lu (2005)]. An exact test procedure can be constructed based on Giri (2004) which is very similar to the conditional approach discussed in Halperin (1961) and Krishnamoorthy and Lu (2005). Our goal here is to provide an exact test and confidence interval for $\mu$, utilizing $\hat{\mu}$ based on the generalized $p$-value approach.

By Cholesky Decomposition, we can find an upper triangular matrix $C$ such that $\Sigma = C'C$. Hence $\Sigma^{-1} = C^{-1}C'^{-1}$. It easy to show that when $\Sigma$ is known, $\hat{\mu}^* \sim N(\mu, \frac{1}{m\Sigma^{-1}})$. Hence

$$Z = \sqrt{n\Sigma^{-1}}(\hat{\mu}^* - \mu) \sim N(0, 1).$$

As $S$ is a symmetric positive definite matrix, we can find an upper triangular matrix $A$ such that $S = A'A$. Let $s$, $a$ be the observed matrix of $S$ and $A$, respectively. We can
represent $S$ as $S = \sum_{\alpha=1}^{n-1} z_{\alpha} z_{\alpha}^T$ where $z_{\alpha} \sim N(0, \Sigma)$. So $C' SC^{-1} = \sum_{\alpha=1}^{n-1} (C' z_{\alpha} z_{\alpha}^T C^{-1})$ where $C' z_{\alpha} \sim N(0, I_k)$ as $\Sigma = C' C$. Then

$$C' A' AC^{-1} = B' B \sim \text{Wishart}(n-1, I_k)$$

where $B = AC^{-1}$. Now from Theorem 3.2.14 in Muirhead (1982), since $B$ is an upper triangular matrix, we can conclude that

$$B_{ii}^2 \sim \chi^2_{n-i} \quad \text{and} \quad B_{ij} \sim N(0,1).$$

Here $B = ((B_{ij}))$ and all $B_{ij}$’s are independent. Following the same technique as before, we can define a generalized pivotal quantity, $T_c$, as

$$T_c = \frac{1' a^{-1} AC^{-1} A' a^{-1} \bar{x}}{1' a^{-1} AC^{-1} A' a^{-1} \bar{x}} - \frac{\sqrt{n} T_{2-1}^2(\mu - \mu)}{\sqrt{n} a^{-1} AC^{-1} A' a^{-1} \bar{x}}$$

$$= \frac{1' a^{-1} BB' a^{-1} \bar{x}}{1' a^{-1} BB' a^{-1} \bar{x}} - \frac{Z}{\sqrt{n} a^{-1} BB' a^{-1} \bar{x}}. \quad (8)$$

It is easy to show that $T_c$ satisfies the conditions of being a pivotal quantity. The distribution of $T_c$ does not depend on any parameter and its observed value is $\mu$. Hence we can use this to construct a generalized confidence interval for $\mu$.

If we write $T = T_c - \mu$, then we can claim by following previous techniques that $T$ is indeed a generalized test variable which can be used to compute the generalized $p$-value.

Consider the following hypotheses:

$$H_0: \mu \leq \mu_0 \quad \text{vs.} \quad H_1: \mu > \mu_0.$$

From $T_c$, we can derive a generalized test variable $T$ as $T = T_c - \mu$. So the first two properties are automatically satisfied. Since distributions of the random variables $V_1, \ldots, V_k, Z$ are free of parameters, it can be shown that $P[T \leq t; \mu | \bar{x}, s] = P[T \leq \mu | \bar{x}, s]$ which is an increasing function of $\mu$. Hence generalized $p$-value can be obtained as

$$p = P[T \leq t; \mu = \mu_0; \bar{x}, s] = P[T \leq \mu_0 | \bar{x}, s].$$

$H_0$ can be rejected if $p < \alpha$. 

9
6 Simulated coverage probabilities and powers

In this section, for two univariate normal populations, we report in Table 6.1 the simulated coverage probabilities (CP), expected lengths (EL), and powers (POW) based on the four estimates of \( \mu \) mentioned in section 1. We also report in Table 6.2 the simulated CP, EL, and POW of the proposed method suggested in section 5, taking a bivariate normal set up.

The following steps were used to simulate coverage probabilities and expected lengths in the univariate case.

Step 1: Generate \( n \) independent normal variables under \( N(0, \sigma_1^2 = 1) \) and \( N(0, \sigma_2^2) \), respectively, for a given value of \( \sigma_2^2 \).

Step 2: The given data set yields \((x_1, \bar{x}_2, v_1, v_2)\). We next generate \( Z \sim N(0, 1) \) and independent chi-squares \((V_i/\sigma_i^2)\). Using these values, we compute the values of \( T_c \) from (4) for the three test procedures based on \( \hat{\mu}_1, \hat{\mu}_3 \) and \( \hat{\mu}_4 \). For the test procedure based on \( \hat{\mu}_2 \), we use (7). This process of generating the values of \( T_c \) is repeated 1000 times for the fixed values of \((x_1, \bar{x}_2, v_1, v_2)\).

Step 3: From the generated values of \( T_c \), we compute \((t_1, t_2)\), the confidence interval for \( \mu \), based on (3), at 95% confidence level.

Step 4: We repeat steps 1 to 3 10000 times, generating different sets of \((x_1, \bar{x}_2, v_1, v_2)\) still under \( N(0, 1) \) and \( N(0, \sigma_2^2) \) for the same value of \( \sigma_2^2 \). In the process, we have generated 10000 confidence intervals \((t_1, t_2)\) for \( \mu \).

Step 5: We compute the percentage of times these intervals contain \( 0 \) [the true value of the parameter] and report this as the simulated coverage probability (CP). We take 95% as targeted confidence coefficient. Obviously, EL is computed by averaging the values of \((t_2 - t_1)\).

Step 6: We repeat steps 1 to 5 for some selected combinations of \( \sigma_2^2 \) and sample size \( n \).

To simulate powers, we assume \( H_0 : \mu = 0 \) against \( H_1 : \mu \neq 0 \), and proceed as follows.

Step 1: Generate \( n \) independent normal variables under \( N(\mu_1, \sigma_1^2 = 1) \) and \( N(\mu_1, \sigma_2^2) \), respectively, for a given value of \( \sigma_2^2 \) where \( \mu_1 \) is the value of \( \mu \) under the alternative (two-
sided). In our simulation, we have taken $\mu_1 = 1$, $\alpha = 0.05$.

**Step 2:** The observed data set yields $(\bar{x}_1, \bar{x}_2, v_1, v_2)$. We next generate $Z \sim N(0, 1)$ and independent chi-squares ($V_i/\sigma^2_i$). Using these values, we compute the values of $T$ [equation (4)] for the four test procedures based on $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$ and $\hat{\mu}_4$. This process of generating the values of $T$ is repeated 1000 times for the fixed values of $(\bar{x}_1, \bar{x}_2, v_1, v_2)$.

**Step 3:** From the generated values of $T$, we compute the generalized p-value based on equation (6) [two sided alternative], taking $\alpha = 0.05$. This is simulated p-value for the corresponding test procedure.

**Step 4:** We repeat steps 1 to 3 10000 times, generating different sets of $(\bar{x}_1, \bar{x}_2, v_1, v_2)$ still under $N(\mu_1, 1)$ and $N(\mu_1, \sigma^2_2)$ for the same value of $\sigma^2_2$. In the process, we have generated 10000 generalized p-values for the relevant test methods.

**Step 5:** We now compute the percentage of times these p-values are less than 0.05, implying rejection of the null hypothesis, and report this as simulated power of the relevant test procedure.

**Step 6:** We repeat steps 1 to 5 for some selected combinations of $\sigma^2_2$ and sample size $n$.

Under the multivariate normal set up, we followed the same procedures to simulate coverage probabilities, expected lengths, and powers as described above.

In step 1, we generate paired samples from a bivariate normal distribution with zero mean vector and certain values of $\rho$ and $\sigma^2_2$. $\sigma^2_1$ is taken to be 1 throughout the process of simulation.

In step 2, the observed samples yield $\bar{x}$ and $s$, the sample mean vector and the sample variance-covariance matrix. The matrix $a$ is obtained by Cholesky decomposition of $s$.

In step 3, we generate $Z$ and $B$ and use equation (8) to compute $T_c$.

The rest of the steps are similar to what we followed in the univariate case. For coverage probabilities (CP), we take the confidence level to be 95% and several combinations of the values of $\rho$, $\sigma^2_2$ and $n$. EL is computed analogously.

For simulation of power, we consider $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$ and take $\mu = 1$ in
step 1 and $\alpha = 0.05$.

<table>
<thead>
<tr>
<th>$\sigma_2^2$</th>
<th>$n$</th>
<th>CP</th>
<th>EL</th>
<th>POW</th>
<th>CP</th>
<th>EL</th>
<th>POW</th>
<th>CP</th>
<th>EL</th>
<th>POW</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5</td>
<td>0.93</td>
<td>1.13</td>
<td>0.91</td>
<td>0.93</td>
<td>1.13</td>
<td>0.91</td>
<td>0.95</td>
<td>1.39</td>
<td>0.80</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.94</td>
<td>0.75</td>
<td>0.99</td>
<td>0.94</td>
<td>0.75</td>
<td>0.99</td>
<td>0.95</td>
<td>0.85</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.95</td>
<td>0.60</td>
<td>1.00</td>
<td>0.95</td>
<td>0.60</td>
<td>1.00</td>
<td>0.95</td>
<td>0.67</td>
<td>0.99</td>
</tr>
</tbody>
</table>

| 1           | 5   | 0.93| 1.36| 0.79| 0.93| 1.38| 0.79| 0.95| 1.52| 0.72|
|             | 10  | 0.94| 0.92| 0.97| 0.94| 0.91| 0.99| 0.95| 0.95| 0.98|
|             | 15  | 0.94| 0.74| 0.99| 0.95| 0.74| 0.99| 0.95| 0.75| 0.99|

| 1.5         | 5   | 0.93| 1.50| 0.72| 0.93| 1.54| 0.70| 0.95| 1.61| 0.68|
|             | 10  | 0.94| 1.01| 0.97| 0.94| 1.01| 0.97| 0.95| 1.02| 0.95|
|             | 15  | 0.94| 0.81| 0.99| 0.95| 0.81| 0.99| 0.96| 0.81| 0.99|

| 2           | 5   | 0.93| 1.60| 0.68| 0.93| 1.66| 0.65| 0.95| 1.67| 0.64|
|             | 10  | 0.94| 1.07| 0.94| 0.94| 1.07| 0.94| 0.95| 1.07| 0.95|
|             | 15  | 0.95| 0.86| 0.99| 0.94| 0.86| 0.97| 0.95| 0.86| 0.99|

It is evident from Table 6.1 that all the four methods are reasonable in terms of CP, EL and POW with method 3 having some edge with respect to size.

We have also carried out a simulation study for the generalized $p$-value method by Krishnamoorthy and Lu (2003). It turns out that this method (KL) has a larger CP and hence a larger EL. For details, we refer to Mitra (2006).
Table 6.2: Simulated CP, EL and POW for the proposed gen-$p$ method

<table>
<thead>
<tr>
<th>$\sigma_2^2$</th>
<th>$\rho$</th>
<th>$n = 5$</th>
<th>$n = 10$</th>
<th>$n = 15$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>CP EL Power</td>
<td>CP EL Power</td>
<td>CP EL Power</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.89 1.60 0.68</td>
<td>0.93 0.97 0.96</td>
<td>0.94 0.76 0.99</td>
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<tr>
<td></td>
<td>0.25</td>
<td>0.88 1.75 0.61</td>
<td>0.93 1.08 0.96</td>
<td>0.93 0.85 0.98</td>
</tr>
<tr>
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<td>0.50</td>
<td>0.86 1.78 0.60</td>
<td>0.91 1.10 0.91</td>
<td>0.92 0.87 0.98</td>
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<tr>
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<td>0.75</td>
<td>0.78 1.45 0.68</td>
<td>0.81 0.90 0.94</td>
<td>0.81 0.71 0.98</td>
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<tr>
<td>1.25</td>
<td>0</td>
<td>0.88 1.69 0.64</td>
<td>0.93 1.03 0.94</td>
<td>0.94 0.81 0.99</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.87 1.88 0.58</td>
<td>0.93 1.16 0.88</td>
<td>0.94 0.91 0.98</td>
</tr>
<tr>
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<td>0.91 1.20 0.86</td>
<td>0.93 0.95 0.96</td>
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<tr>
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<td>0.78 1.58 0.64</td>
<td>0.80 0.99 0.80</td>
<td>0.80 0.78 0.97</td>
</tr>
<tr>
<td>1.50</td>
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<td>0.87 1.78 0.63</td>
<td>0.93 1.08 0.92</td>
<td>0.94 0.84 0.99</td>
</tr>
<tr>
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<td>0.87 2.00 0.55</td>
<td>0.93 1.23 0.86</td>
<td>0.94 0.96 0.97</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.85 2.06 0.52</td>
<td>0.93 1.29 0.81</td>
<td>0.92 1.03 0.94</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.76 1.69 0.60</td>
<td>0.79 1.07 0.85</td>
<td>0.80 0.85 0.95</td>
</tr>
<tr>
<td>1.75</td>
<td>0</td>
<td>0.87 1.85 0.60</td>
<td>0.93 1.11 0.90</td>
<td>0.94 0.87 0.98</td>
</tr>
<tr>
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<td>0.25</td>
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<td>0.93 1.28 0.82</td>
<td>0.94 1.01 0.95</td>
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<tr>
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<td>0.84 2.18 0.50</td>
<td>0.91 1.38 0.77</td>
<td>0.93 1.09 0.92</td>
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<td>0.78 0.92 0.81</td>
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<tr>
<td>2</td>
<td>0</td>
<td>0.87 1.92 0.59</td>
<td>0.92 1.15 0.88</td>
<td>0.93 0.89 0.98</td>
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<tr>
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<td>0.86 2.19 0.51</td>
<td>0.93 1.34 0.81</td>
<td>0.94 1.05 0.94</td>
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<tr>
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<td>0.75</td>
<td>0.75 1.91 0.55</td>
<td>0.78 1.22 0.78</td>
<td>0.78 0.98 0.89</td>
</tr>
</tbody>
</table>

It is observed from the above simulation study that the method based on generalized $p$-value is performing quite well in terms of coverage probability when $n$ is 10 or more and correlation $\rho$ is not more than 0.50, although the coverage probability for $n = 5$ is far below
95%. This method may not perform well in higher dimensions.

7 Application

We conclude this paper with an application of the proposed methods discussed in sections 3 and 4 to a suitably truncated data set derived from Eberhardt et. al. (1989), and reproduced below in Table 6.3. Details of analysis given in Table 6.4 suggest that method 4 performs the best in terms of expected length.

Table 6.3: Truncated data set to apply gen $p$-value methods

<table>
<thead>
<tr>
<th>Methods</th>
<th>$n_i$</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atomic absorption spectrometry</td>
<td>8</td>
<td>105.0</td>
<td>85.711</td>
</tr>
<tr>
<td>Isotope dilution mass spectrometry</td>
<td>8</td>
<td>113.25</td>
<td>33.640</td>
</tr>
</tbody>
</table>

Table 6.4: Comparison of generalized $p$-value methods

<table>
<thead>
<tr>
<th>GP Methods</th>
<th>CI</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(105.86, 115.00)</td>
<td>9.14</td>
</tr>
<tr>
<td>2</td>
<td>(106.35, 115.18)</td>
<td>8.83</td>
</tr>
<tr>
<td>3</td>
<td>(104.35, 113.63)</td>
<td>9.28</td>
</tr>
<tr>
<td>4</td>
<td>(105.73, 114.17)</td>
<td>8.44</td>
</tr>
<tr>
<td>KL</td>
<td>(105.40, 115.59)</td>
<td>10.19</td>
</tr>
</tbody>
</table>

Acknowledgment. Our sincere thanks are due to the referee for some very helpful comments which improved the presentation of the paper.

References


