Modeling the geometric features and investigating electrical properties of dendrites in a fish thalamic neuron

by

Wojciech Krzyzanski¹ and Jonathan Bell²

¹ Department of Pharmaceutical Sciences, SUNY at Buffalo
² Department of Mathematics & Statistics, UMBC

Abstract: A certain pacific fish has a thalamic nerve cell with a unique dendritic geometry. Instead of the cell having a branching dendritic structure, the large cell of the corpus glomerulosum, a part of the thalamus considered associated with the fish’s vision, has generally a single dendritic stalk with a large bulbous tip. We formulate a cable theory model that incorporates the geometry, and then we solve the problem for a single localized synaptic current source at the bulb. From the solution representation we numerically examine the characteristics of the postsynaptic potential due to the presence of the bulbous tip.

Keywords: dendritic bulb, corpus glomerulosum, cable theory, eigenvalue problem, Legendre equation, Legendre functions, Greens’ function, Galerkin approximation

1. Introduction

There may be significant information processing going on even at the cellular level in CNS neurons [10], but most neurons have very complicated geometrical features, including complex dendritic arborizations, varicosities, and many different types of ion conductances and synaptic connections that are nonuniformly distributed. This makes analysis of various morphological features difficult to assess. Yet dendritic geometry is an important aspect in characterizing properties and function of nerve cells.

Tsutsui, et al [12,13] have focused on studying the morphology and physiology of the corpus glomerulosum (CG) in a teleost fish, Stephanoplepis cirrifer, which has a well-defined architecture. The CG is an expansive nucleus that might be involved in vision-related information processing, and in the control of hypothalamic functions [12]. In the fully laminated type CG there are two main cell types, "large" cells, whose soma diameter is 20-30 µm, and "small" cells, with soma diameter being 7-10 µm. An additional feature of the large cells is the giant tip bulb that generally has diameter greater than 50 µm. That is, these cells, which are the subject of this paper, are characterized by one, occasionally two, dendrites with huge spherical bulbous endings rather than branching structures.

Our interest here concerns the effect the large bulbous dendritic ending has on the postsynaptic potential reaching the cell body. This enlargement is not found in the brain cells of other vertebrates. The larger the area of the postsynaptic membrane the larger the number of postsynaptic terminals the cell can receive. But the larger the postsynaptic membrane the lower the expected input resistance, which decreases the amplitude of the
postsynaptic potential contributed at each synaptic terminal. Tsutsui speculates that the large dendritic tip is specialized for "averaging" synaptic inputs from large numbers of corticalis axons making connections to the large CG cell.

Because of the lack of a complicated dendritic branching structure, Tsutsui and Oka [13] constructed a four compartment model for the large CG cell (axon, soma, dendritic stalk, dendritic bulb) and used the simulation program NEURON [5] to study the effect of the bulb size on properties of the potential at the soma. They demonstrate a geometric "boosting" effect, i.e. passive propagation of the synaptic inputs from the dendritic bulb to the soma through the dendritic stalk is less attenuated in cells with larger bulbs. They also made a few preliminary simulations using active Hodgkin-Huxley type sodium and potassium conductances in the bulb, stalk, soma, and axon, and noted with a normal length stalk the size of the bulb made little effect on the firing characteristics at the soma. At this time there is little information on the distribution of active conductances in these CG cells. In fact, they admit that there appears from the experimental results that there are no active conductances in the dendritic bulb or stalk. But since the cell itself fires single sodium spikes, the soma and axon probably are the location for these nonlinear voltage channels.

In this paper we reexamine the modeling approach to the large CG cell dendrite. Instead of doing a multi-compartment simulation, we consider instead a continuous cable theory model that accounts for the regular, but nonuniform, geometry of the dendrite. This also allows us to analytically solve the partial differential equation problem via classical methods, and then explore changes in behavior with regard to changing geometric parameters. The basic model is a current balance equation for the transmembrane potential (equation (1) below) that is derived from a circuit model, is shown in the Appendix. Our goal is to see how the postsynaptic potential near the soma is affected by a variety of parameters associated with the dendritic bulb and stalk. So there will not be a need to include the soma in the present study. Thus, the description of the dendritic radius to our "lollipop" model of the CG cell dendrite is given in Figure 1.

In the next section we describe the model equations and constraints, then nondimensionalize the equations. The cell is stimulated at a single, localized point source on the bulb, which models the use of a micropipette. This leads to an eigenvalue problem, actually a coupled system because of the different operator that arises for the bulb, versus the operator for the dendritic stalk. The eigenvalue problem is solved in section three and then used in the integral representation of the solution in section four. In section five we use the solution to compute peak and time to peak of the postsynaptic potential, and the attenuation of the potential from the bulb to near the soma. In particular, we are interested in how the synaptic density on the bulb affects postsynaptic potential attenuation. In the final section we draw some conclusions from the work.
Figure 1: The variable cable radius used in the paper, with radius \( a(x) \) given by (2). The Model just involves the dendritic bulb \((-\alpha \leq x \leq x_0)\) and stalk \((x_0 \leq x \leq \ell)\), with \(x = \ell\) being proximal to cell soma.

2. The Model

In the Appendix we give a brief derivation of the cable model used in this paper, namely

\[
C_m \frac{\partial V}{\partial t} + I_{\text{ion}}(v) = \frac{1}{2R_a a(1 + (da/dx)^2)} \frac{\partial}{\partial x} \left[a^2 \frac{\partial V}{\partial x}\right] - I_{\text{syn}}(x,t) \quad (1)
\]

Here \(C_m\) and \(R_i\) are membrane capacitance and axoplasmic resistivity, respectively. Since no active conductances have been found in the large CG cell's dendritic stalk or tip [13], we consider a passive current-voltage relation here, namely \(I_{\text{ion}}(v) = (v - v_{\text{rest}})/R_m\), where \(R_m\) is the membrane resistivity and \(v_{\text{rest}}\) is the membrane resting potential. The \(I_{\text{syn}}\) term on the right-hand side of (1) represents any external current sources, like synaptic junctions. We assume that the synaptic current density is determined by the reversal potential \(E_{\text{syn}}\) and the conductivity \(g_{\text{syn}}(x,t)\), so from the Appendix

\[
I_{\text{syn}}(x,t) = i_{\text{ext}} / dA / dx = g_{\text{syn}}(x,t)(v - E_{\text{syn}}) \quad .
\]

Additionally, we assume that the current is localized at a single point \(x = x_{\text{syn}}\) and the conductivity is controlled by the standard \(\alpha\)-function [10]. From Figure 1 we let

\[
a(x) = \begin{cases} \sqrt{\alpha^2 - x^2} & -\alpha \leq x < x_0 = \sqrt{\alpha^2 - \beta^2} \\ \beta & x_0 \leq x \leq \ell \end{cases} \quad .
\]

Substituting (2) into equation (1), we obtain
\[
C_m \frac{\partial v}{\partial t} + \frac{v - v_{rest}}{R_m} = \begin{cases}
\frac{1}{2R_l \alpha} \frac{\partial}{\partial x} \left( (\alpha^2 - x^2) \frac{\partial v}{\partial x} \right) - I_{syn}(x,t) & -\alpha < x < x_0 \\
\frac{\beta}{2R_l} \frac{\partial^2 v}{\partial x^2} & x_0 < x < l
\end{cases}
\] 

(3)

In particular, the synaptic conductance density becomes

\[
g_{syn}(x,t) = \frac{\bar{g}_{syn} t / \tau_{syn} e^{-t/\tau_{syn}}}{2\pi \alpha} \delta(x - x_{syn}).
\]

To nondimensionalize (3), let \( V_{ch} \) be some characteristic potential, \( \tau_m \) be the membrane time constant \( R_m C_m \), \( \lambda \) be the space constant for some characteristic length \( L \), i.e. \( \lambda = \sqrt{LR_m / 2R_l} \). Then letting

\[
\tilde{t} = t / \tau_m, \tilde{x} = x / \lambda, \tilde{v} = (v - v_{rest}) / V_{ch},
\]

we obtain

\[
\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{v} = \begin{cases}
\frac{1}{\alpha L} \frac{\partial}{\partial \tilde{x}} \left( (\alpha^2 - \lambda^2 \tilde{x}^2) \frac{\partial \tilde{v}}{\partial \tilde{x}} \right) - i_{syn}(\tilde{x}, \tilde{t}) & -\bar{\alpha} < \tilde{x} < \bar{x}_0 \\
\frac{\beta}{L} \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} & \bar{x}_0 < \tilde{x} < \bar{\ell}
\end{cases}
\]

Here \( i_{syn}(\tilde{x}, \tilde{t}) = R_m I_{syn}(\lambda \tilde{x}, \tau_m \tilde{t}) / V_{ch} \), and \((\bar{\alpha}, \bar{x}_0, \bar{\ell}) = (\alpha, x_0, \ell) / \lambda \). Typically the characteristic length scale is associated with the fiber radius, so let \( L = \beta \); if we pull out the \( \lambda^2 \) and define \( \delta \equiv \lambda / \bar{\alpha} \beta \), and then drop the tilde notation for convenience, to obtain

\[
\frac{\partial v}{\partial t} + v = \frac{\partial}{\partial x} \left( \Lambda(x) \frac{\partial v}{\partial x} \right) - i_{syn}(x,t), \quad \text{for } -\alpha < x < \ell \text{ and } x \neq x_0
\]

(4)

where \( \Lambda(x) = \delta(x^2 - \alpha^2) \) for \(-\alpha < x < x_0\), and unity otherwise. This is the model system we work with below. The boundary condition we use at \( x = \ell \) is "sealed-end" condition, i.e. no leak of longitudinal current:

\[
\frac{\partial v}{\partial x}(\ell,t) = 0, \quad 0 < t < T.
\]

(5a)

Due to the singularity of the diffusion operator in (4) at \( x = -\alpha \), the sealed end boundary condition at this end is weakened to
\[ \Lambda(x) \frac{\partial V}{\partial x}(x, t) = o(1) \quad \text{as} \quad x \to - \alpha^+ , \quad 0 < t < T. \quad (5b) \]

Since the cable equation (4) does not hold at \( x = x_0 \), the continuity conditions are imposed on the membrane potential

\[ v(x_0^-, t) = v(x_0^+, t), \quad 0 < t < T, \quad (5c) \]

and the current

\[ \Lambda(x_0^+) \frac{\partial V}{\partial x}(x_0^+, t) = \Lambda(x_0^-) \frac{\partial V}{\partial x}(x_0^-, t), \quad 0 < t < T. \quad (5d) \]

For now we leave the initial voltage distribution arbitrary, that is, \( v(x, 0) = v_0(t) \). In the simulations below we consider \( v_0(t) \equiv 0 \), the rest potential for the dendrite. The dimensionless density of synaptic current becomes

\[ i_{\text{syn}}(x, t) = g_s(x, t)(v - E_s), \quad g_s(x, t) = \frac{\overline{g_s}}{\tau_s} \frac{t}{\tau_s} e^{-t/\tau_s} \delta(x - x_s), \quad -\alpha < x_s < x_0. \quad (6) \]

and \( i_{\text{syn}}(x, t) = 0 \) for \( x_0 < x < \ell \). Here

\[ \overline{g_s} = \frac{g_{\text{syn}} R_m}{2\pi \alpha \lambda^2}, \quad \tau_s = \tau_{\text{syn}}/\tau_m, \quad E_s = (E_{\text{syn}} - v_{\text{rest}})/V_{\text{ch}}, \quad \text{and} \quad x_s = x_{\text{syn}}/\lambda, \]

and \( i_{\text{syn}}(x, t) = 0 \) for \( x_0 < x < \ell \).

### 3. The Eigenvalue Problem

We will solve the problem (4)-(6) using the eigenfunction expansion technique (Evans [4]). The corresponding eigenvalue problem can be formulated as follows:

\[ -\frac{d}{dx} \left( \Lambda(x) \frac{d\phi}{dx} \right) + \phi = \lambda \phi, \quad \text{for} \quad -\alpha < x < \ell \quad \text{and} \quad x \neq x_0 \quad (7) \]

The boundary conditions are

\[ \Lambda(x) \frac{d\phi}{dx}(x) = o(1) \quad \text{as} \quad x \to - \alpha^+ \quad (8) \]

and

\[ \frac{d\phi}{dx}(\ell) = 0. \quad (9) \]

Since the function \( \Lambda(x) \) in (7) lacks continuity in its derivative at \( x = x_0 \), the gluing conditions get translated to \( \phi(x) \) as

\[ \phi(x_0^-) = \phi(x_0^+) \quad (10) \]

and
\[ \delta \beta^2 \frac{d \phi}{dx}(x_0-) = \frac{d \phi}{dx}(x_0+) \] \hspace{1cm} (11)

Multiplying both sides of (7) by \( \phi(x) \) and integrating from \(-\alpha \) to \( \ell \) yields

\[ \lambda \geq 1 \] \hspace{1cm} (12)

Changing the variable \( x = \alpha z \) and \( \phi(z) = \phi(\alpha z) \) reduces (7) to the Legendre equation

\[ (1 - z^2) \frac{d^2 \phi}{dz^2} - 2z \frac{d \phi}{dz} + \nu(\nu + 1)\phi = 0, \quad \text{for} \quad -1 < z < x_0/\alpha \] \hspace{1cm} (13)

and

\[ -\frac{1}{\alpha^2} \frac{d^2 \phi}{dz^2} + \phi = \lambda \phi, \quad \text{for} \quad x_0/\alpha < z < \ell/\alpha \] \hspace{1cm} (14)

where

\[ \nu(\nu + 1) = \frac{\lambda - 1}{\delta} \] \hspace{1cm} (15)

The Legendre equation (13) remains unchanged if \( \nu \) is replaced by \(-\nu-1\). Therefore, without loss of generality, we select the nonnegative root of (15) \( \nu = (-1 + \sqrt{1 + 4(\lambda - 1)/\delta})/2 \) to determine two linearly independent solutions of (13), \( P_\nu(-z) \) and \( Q_\nu(-z) \), where \( P_\nu \) is the Legendre function, and \( Q_\nu \) is the Legendre associated function [1]. The negative sign is introduced on purpose, since then \( P_\nu(-z) \) is an analytic function at \( z = -1 \) and \( Q_\nu(-z) \) is singular at \( z = -1 \) with the following dominant term [3]

\[ Q_\nu(-z) = -\frac{1}{2} \ln \left( \frac{1 + z}{2} \right) + O(1) \quad \text{as} \quad z \to -1 + \] \hspace{1cm} (16)

Since

\[ P_\nu(z) \frac{dQ_\nu}{dz}(z) - Q_\nu(z) \frac{dP_\nu}{dz}(z) = \frac{1}{1 - z^2}, \quad -1 < z < 1 \] \hspace{1cm} (17)

and \( P_\nu(1) = 1 \) and \( P_\nu'(1) = \nu(\nu + 1) \) [3], there exists a real constant \( C_1 \) such that

\[ \phi(z) = C_1 P_\nu(-z), \quad \text{for} \quad -1 < z < x_0/\alpha \] \hspace{1cm} (18)

Since \( \lambda \geq 1 \), the solution to (14) with the boundary condition (9) has the form

\[ \phi(z) = C_2 \cos \left( \sqrt{\lambda - 1}(\alpha z - \ell) \right) \] \hspace{1cm} (19)

for some constant \( C_2 \). The gluing conditions (10) and (11) yield the following system of linear equations for \( C_1 \) and \( C_2 \).
\begin{equation}
C_1 P_\nu \left( -x_0 / \alpha \right) - C_2 \cos \left( \sqrt{\nu(v+1)\delta} (x_0 - \ell) \right) = 0 \tag{20}
\end{equation}

\begin{equation}
- C_1 \delta (\alpha^2 - x_0^2) P'_\nu \left( -x_0 / \alpha \right) + C_2 \alpha \sqrt{\nu(v+1)\delta} \sin \left( \sqrt{\nu(v+1)\delta} (x_0 - \ell) \right) = 0 \tag{21}
\end{equation}

A non-zero solution exists only if the determinant of the matrix for this system is zero. This yields the transcendental equation for the boundary value problem (7)-(8), namely

\begin{equation}
\alpha \sqrt{\nu(v+1)\delta} \sin \left( \sqrt{\nu(v+1)\delta} (x_0 - \ell) \right) P_\nu \left( -x_0 / \alpha \right) - \left( \beta / \alpha \right) \cos \left( \sqrt{\nu(v+1)\delta} (x_0 - \ell) \right) P'_\nu \left( -x_0 / \alpha \right) = 0 \tag{22}
\end{equation}

The plots of the left-hand side of (22) is presented in Fig. 2.

![Figure 2](image_url)

**Figure 2.** Roots of equation (22). Each intercept of the graph of the left-hand side of (22) with the horizontal axis determine $\nu$, and consequently an eigenvalue of the problem (4)-(6). A Fortran subroutine hypser.for [9] was used to calculate $P_\nu(x)$ and $P'_\nu(x)$. The parameter $\alpha = 20 \text{ } \mu\text{m}$ and other parameters from Table 1 were nondimensionalized and used for simulations.

All non-negative solutions $\nu \geq 0$ of (22) determine via (15) the eigenvalues for the problem (7)-(8)

\begin{equation}
\lambda = 1 + \delta \nu(v+1) \tag{23}
\end{equation}

One can verify that there exists a constant $C$ such that for $\nu$ that
\[
\phi(x) = \begin{cases} 
C \cos\left(\sqrt{v(v+1)}(x_0 - \ell)\right) \mathcal{P}_v(-x / \alpha), & \text{if } -\alpha \leq x \leq x_0 \\
C \cos\left(\sqrt{v(v+1)}(x - \ell)\right) \mathcal{P}_v(-x_0 / \alpha), & \text{if } x_0 \leq x \leq \ell 
\end{cases}
\] (24)

Behavior of the left-hand side of (22) as \(v \to \infty\) implies that the set of solutions is countable and unbound (Fig. 2). They can be numbered in the increasing order

\(0 = \nu_1 < \nu_2 < \ldots\) and \(\nu_n \to \infty\) as \(n \to \infty\) (25)

Subsequently, there are infinitely many eigenvalues \(\lambda_n, n = 1, 2, \ldots\) that approach infinity as \(n\) becomes large. For each \(n = 1, 2, \ldots\) there corresponds a unique eigenfunction \(e_n(x)\) such that

\[
e_n(x) = \begin{cases} 
K_n \cos\left(\sqrt{\frac{\lambda_n - 1}{\lambda_n}}(x_0 - \ell)\right) \mathcal{P}_v(-x / \alpha), & \text{if } -\alpha \leq x \leq x_0 \\
K_n \cos\left(\sqrt{\frac{\lambda_n - 1}{\lambda_n}}(x - \ell)\right) \mathcal{P}_v(-x_0 / \alpha), & \text{if } x_0 \leq x \leq \ell 
\end{cases}
\] (26)

where the constant \(K_n > 0\) is determined by the normalizing condition

\[
\int_{-\alpha}^{\ell} e_n(x)^2 \, dx = 1
\] (29)

For the smallest eigenvalue \(\lambda_1 = 1\), (28) and (29) imply that \(e_1\) is the constant function

\[
e_1(x) = \frac{1}{\sqrt{\ell + \alpha}}
\] . (30)

In Fig. 3 there are shown the graphs of the first five eigenfunctions.

Later the following two properties of eigenfunctions and eigenvalues will be important. For any small \(\varepsilon > 0\), there exists a constant \(M > 0\) such for any \(n = 1, 2, \ldots\) and any \(-\alpha + \varepsilon \leq x \leq \ell, x \neq x_0\)

\[
|e_n(x)| \leq M \quad \text{and} \quad \left|\frac{de_n}{dx}(x)\right| \leq M\sqrt{\lambda_n}
\] (31)

There is a constant \(d > 0\) and an integer \(K \geq 0\) such that for any \(n = 1, 2, \ldots\)

\[
\nu_{n+K} \geq \nu_n + d
\] (32)

We leave the proof of (31) and (32) to the reader. Fig. 2 shows that \(K\) can be positive.
4. Integral Representation of the Solution

The theory of compact and self-adjoint operators implies that the family of eigenfunctions \( \{e_n\}_{n=1,2,\ldots} \) is orthonormal and complete [14]. Using (31) and (32) one can verify by direct calculation that

\[
G(x, y, t) = \sum_{n=1}^{\infty} e_n(x)e_n(y)e^{-\lambda_n t}, \quad -\alpha < x, y \leq \ell, \ t > 0
\]  

(33)

is the Greens function for the problem (4)-(5) [6]. Representing the solution of this problem by the means of Greens function, one can transform it to the following integral form

\[
v(x, t) = \int_{0}^{\ell} \int_{-\alpha}^{\ell} g_{\text{syn}}(y, z)(E_s - v(y, z))G(x, y, t - z)dydz
\]  

(34)

Given the explicit form of \( g_{\text{syn}}(x, t) \) in (6), one can further simplify (34) to

\[
v(x, t) = \int_{0}^{\ell} g(z)(E_s - v(x, z))G(x, x, t - z)dz
\]  

(35)

where

Figure 3. The graphs of the eigenfunctions \( e_n(x) \), \( n = 1, \ldots, 5 \). The parameters used for simulations are same as in Fig. 2.
\[ g(t) = \frac{\bar{g}_s}{\tau_s} \frac{t}{e^{t/\tau_s}} = g_0 t e^{-t/\tau_s}. \]  

(36)

In order to obtain the complete representation of the solution of (4)-(5) one needs to know \( v(x_s,t) \). This can be determined from (34) by evaluating the integral at \( x = x_s \) and solving the following Volterra integral equation of the second type for \( w(t) = v(x_s,t) \):

\[ w(t) = \int_0^t g(z)(E_s - w(z))G(x_s,x_s,t-z)dz. \]  

(37)

The kernel is singular at \( t = z \). One can apply (31) and (33) to estimate this singularity as \( O(1/(t-z)^\gamma) \) as \( z \to t^+ \), where \( \gamma > \frac{1}{2} \) (see Appendix). This guarantees existence of a unique solution [7]. Below we present an approach applying the Galerkin approximation of the solution using the eigenfunction expansion representation of \( G \) in (33) [11]. For a given positive integer \( N \geq 1 \), let

\[ G_N(x,y,t) = \sum_{n=1}^{N} e_n(x)e_n(y)e^{-\lambda_n t}. \]  

(38)

be an approximation to the Green’s function \( G \). Let \( w_N \) denote the solution of (37) with \( G_N \) substituted for \( G \) and \( v_N \) be represented by (35) with \( w_N \) and \( G_N \) instead of \( w \) and \( G \). In Appendix B we show that for sufficiently small \( \varepsilon > 0 \)

\[ v_N(x,t) \to v(x,t) \text{ as } N \to \infty \text{ uniformly in } -\alpha + \varepsilon \leq x, y \leq \ell, 0 \leq t \leq T \]  

(39)

Let

\[ u^N_n(t) = \int_0^1 g(z)(E_s - w_N(z))e_n(x_s)^2 e^{-\lambda_n (t-z)}dz \quad n = 1, \ldots, N \]  

(40)

so that from (40), (38) and (37)

\[ w_N = \sum_{n=1}^{N} u^N_n \]  

(41)

Since \( G_N \) as a function of time is a linear combination of exponential functions, one can differentiate the integral equation for \( w_N \) and transform it to the following system of linear ODEs

\[ \frac{du^N_n}{dt} = g(t)\left(E_s - \sum_{k=1}^{N} u^N_k\right)e_n(x_s)^2 - \lambda_n u^N_n, \]  

\[ n = 1, \ldots, N \]  

(42)

with the zero initial conditions. Notice that where
\[ v_N(x, t) = \sum_{n=1}^{N} \frac{e_n(x)u_n^N(t)}{e_n(x_s)} \] 

For \( N = 1 \), the solution of (40) yields \( w_1 \) given by

\[ w_1(t) = \frac{E_s}{\ell + \alpha} \int_0^t g(z) \exp \left( z - t + \int_0^t g(p) dp / (\ell + \alpha) \right) dz. \]

If we define \( f(z, t) = z - t(1 - 1/\tau_z) + \frac{g_0 \tau_z}{\ell + \alpha} \{ze^{-z/\tau_z} - te^{-t/\tau_z} + \tau_z[e^{-z/\tau_z} - e^{-t/\tau_z}] \}, \),

Then we can write

\[ v_1(x, t) = -\frac{g_0 E_x}{\ell + \alpha} \int_0^t \int z e^{z(1-1/\tau_z)-t} \{1 + \frac{g_0}{\ell + \alpha} \int z' e^{f(z', t)} dz' \} dz. \] 

5. Discussion of Numerical Results

The membrane electrical and geometrical parameters were obtained from [13] and they are listed in Table 1. The eigenvalues were calculated by finding the roots of (22). The bisection method was used as implemented in the Fortran subroutine rtbis.for [9]. The Legendre functions \( P_n(x) \) and their derivatives were evaluated by means of the hypergeometric functions using subroutine hypgeo.for [9]. The eigenfunctions were constructed from (26) where the constant \( K_n \) were determined by the numerical integration. The number \( N = 200 \) of terms in the approximation \( v_N(x, t) \) of the solution \( v(x, t) \) proved to be sufficient. The system of differential equations (42) was solved by the fourth-order Runge-Kutta method [9, 8]. The function \( v_N(x, t) \) was calculated from (41). The propagation of post-synaptic potential from the tip to the soma through the stalk was analyzed by simulation of the time course of the membrane at the current source, at the
beginning of the stalk, and at

Figure 4. The membrane potential at the tip for two different sizes of the bulb following synaptic excitation. The synaptic conductances were same for both bulbs and the synapse was located the middle of the bulb ($x_s = 0$). The tip size was $\alpha = 20 \, \mu m$ and the maximum synaptic conductance $\bar{g}_{\text{syn}} = 0.6 \, \mu S$. The rest potential was set to 0. The remaining parameters were as listed in Table 1.

The soma (Fig. 4) as well as distribution of the membrane potential along the nerve at various times (Fig. 5). Following the synaptic stimulus described by the alpha function, the membrane potential at each point resembled the same pattern with rapid increase and slower return to the resting state. However, the decline time course had a different from the alpha-shape biphasic profile with the rapid drop and a slow return phases. The magnitude of the rapid phase decreased for along the stalk to vanish at the soma. The slow return phase was same for at all locations and started at about 2 msec. After this time the distribution of the potential along the nerve was homogeneous as seen in Fig. 5. The peak time increased with the distance from the synaptic input location, which was the consequence of the potential propagation. Similarly, the potential peaks were lower for points more distant from the synapse due to dissipation of the propagating potential. The membrane potential was highest at the synaptic input and diminished along the tip and stalk with the lowest value at the soma. The differences between potentials at various points of the tip were minimal.
Figure 5. The distribution of the membrane potential along the nerve at various times. The parameters were as in Fig. 4.

The peak membrane potentials at the some as well as peak times were simulated as a function of the tip size where $\alpha$ varied from $2 \, \mu m (=\beta)$ to $40 \, \mu m$ (Fig. 6). The maximum synaptic conductance $\bar{g}_{syn}$ were adjusted for the tip size to keep the maximum synaptic conductance per tip surface area constant. The peak potential increased less than proportionally to the tip size. The peak time increased linearly with the tip size but the increase was incremental.
Figure 6. The peak potential at soma ($x = l$) (thick line) and corresponding peak time (thin line) as functions of the tip size. For simulations $\alpha$ varied from 2 $\mu$m to 40 $\mu$m and the maximum synaptic conductance per tip surface area were constant ($\bar{g}_{\text{syn}}/(2\pi\alpha(x_0)) \equiv 119.4$ pS/$\mu$m$^2$). The remaining parameters were as listed in Table 1. The irregularities of the peak time vs. $\alpha$ curve are due to relatively high increment in $\alpha$ ($\Delta \alpha = 1$ $\mu$m) used to generate this plot when compared to change in the peak time value.

7. Conclusions

Tsutsui et al [13] used the NEURON program [5] to study the effects of the dendritic tip geometry on the electrical properties of teleostat thalamic neurons. NEURON is designed around the notion of continuous cable "sections" which are connected together to form any kind of branched cable and which are endowed with properties which vary continuously with position along the section [5]. This compartmental structure lacks of effects of continuous change of dendrite diameter within each section that can effect spatial membrane distribution. This might be particularly important when the large bulb of the teleostat thalamic neuron is approximated by a cylinder. We applied the cable equation for the nerves with continuously varying diameter where the tip was represented by a sphere. Are simulations of the membrane potential done for identical electrical and geometrical nerve characteristics yielded similar results for the time behavior and the effect of the tip size. Additionally, we were able to simulate the spatial distribution of the membrane potential including the points of the most abrupt change in the dendrite
diameter. Our findings for the assumed settings did not show a noticeable change of the potential along the tip, but provided a physiologically more relevant boundary conditions for the potential at the stalk-tip interface.

References


Appendix: Cable Model with Variable Radius

In this appendix we notationally suppress the time dependence of the time variable for convenience and develop the model from the circuit model in Figure A1. By Kirchhoff laws,
\[ v_i(x) - v_i(x + h) = \frac{h}{r_i(x + h)} i_i(x + h) \quad v_e(x) - v_e(x + h) = \frac{h}{r_e(x + h)} i_e(x + h). \]

If we divide by \( h \) and let \( h \to 0 \), we obtain the differential form of the expressions, namely

\[ -\frac{\partial v_i}{\partial x}(x) = r_i(x) i_i(x), \quad -\frac{\partial v_e}{\partial x}(x) = r_e(x) i_e(x). \]

If we write \( v = v_i - v_e \) for the transmembrane potential, then \( \frac{\partial v}{\partial x} = -r_i i_i + r_e i_e \). Here \( r_i, r_e \) are internal and external resistances per unit length, so \( r_i = R_i / A_i \). \( R_i \) is the axoplasmic resistivity, and \( A_i \) is the cross-sectional area of the cylinder, so with \( a \) being cylinder radius, and

\[ R_i = \frac{\pi a^2 r_i}. \quad (A.1) \]

Similarly, \( r_e = R_e / A_e \), but because the extracellular space is considered a large bath, \( A_e \gg 1 \), so \( r_e \ll 1 \). That is, for all intents and purposes, \( r_e = 0 \), so let

\[ \frac{\partial v}{\partial x} = -r_i i_i. \quad (A.2) \]

Again, by Kirchhoff's laws,

\[ i_i(x) = i_i(x + h) + h i_m(x) + h i_{ext}(x), \quad i_e(x) + h i_m(x) + h i_{ext}(x) = i_e(x + h), \]

which leads to the differential forms

\[ i_m(x) + i_{ext}(x) = -\frac{\partial i_i}{\partial x}(x) = -\frac{\partial i_e}{\partial x}(x). \quad (A.3) \]

Now the membrane current density \( i_m \) is a sum of capacitance current and ionic current densities, i.e.

\[ i_m = (C_m \frac{\partial v}{\partial t} + I_{ion}) \frac{dA}{dx} \quad (A.4) \]

where \( dA/dx \) is the change in membrane area with increase in \( x \). Normally, for the cylinder cable with constant \( a \), it is taken to be \( 2\pi a \). But for variable \( a = a(x) \),

\[ \frac{dA}{dx} = 2\pi a \sqrt{1 + \left(\frac{da}{dx}\right)^2}. \quad (A.5) \]

by the Pythagorean theorem. Often the gradient of \( a \) is assumed small, so the radical factor is considered negligible. Combining the expressions (A.1)-(A.5) gives
\[
(C_m \frac{\partial v}{\partial t} + I_{ion})2\pi a \sqrt{1 + \left( \frac{da}{dx} \right)^2} + i_{ext} = \frac{\partial}{\partial x} \left( \frac{\pi a^2}{R_i} \frac{\partial v}{\partial x} \right)
\]

or
\[
C_m \frac{\partial v}{\partial t} + I_{ion} = \frac{1}{2R_i a \sqrt{1 + \left( \frac{da}{dx} \right)^2}} \left\{ \frac{\partial}{\partial x} \left( a^2 \frac{\partial v}{\partial x} \right) - i_{ext} \right\} . \tag{A.6}
\]

Figure A1: Electric circuit model of the cable membrane.

**Appendix: Uniform Convergence of the Approximate Solutions**

First we show that for any in - \( \alpha < x, y \leq \ell \) and \( \alpha > \frac{1}{2} \),

\[
G(x, y, t) = O\left( \frac{1}{t^\alpha} \right) \quad \text{for} \quad 0 < t < T \tag{A.7}
\]

Since

\[
t^\alpha e^{-\lambda_n t} \leq \frac{\alpha^\alpha e^{-\alpha}}{\lambda_n^\alpha}
\]

and every \( |e_n(x)| \) is bounded by the constant \( M (31) \), the definition (33) of \( G(x, y, t) \) implies

\[
\left| t^\alpha G(x, y, t) \right| \leq M^2 \alpha^\alpha e^{-\alpha} \sum_{n=1}^\infty \frac{1}{\lambda_n^\alpha} \tag{A.8}
\]

From (33) one can conclude that \( \lambda_n = O(1/n^2) \) as \( n \to \infty \). Hence the series in (A8) converges for any \( \alpha > \frac{1}{2} \) and (A7) follows. Next, we will use a generalized form of the Gronwall’s inequality for weakly singular kernels of the Volterra type.
Lemma. Let \( \xi(t) \) and \( \theta(t) \) be continuous on the interval \([0, T]\) and let \( K(t,z) \) be continuous and nonnegative on the triangle \( 0 \leq z \leq t \leq T \). If

\[
\xi(t) \leq \theta(t) + \int_0^t \frac{K(t,z)\xi(z)dz}{(t-z)^\alpha}, \quad 0 \leq t \leq T
\]  

(A.9)

for a constant \( 0 \leq \alpha < 1 \), then

\[
\xi(t) \leq \theta(t) + \int_0^t R(t,z)\theta(z)dz, \quad 0 \leq t \leq T
\]  

(A.10)

where \( R(t,z) \) is the resolvent kernel and \( R(t,z) - K(t,z)/(t-z)^\alpha \) is continuous and nonnegative on the triangle \( 0 \leq z \leq t \leq T \).

Proof of the lemma is presented in [2] for \( \alpha = 1 \). The singular case \( 0 < \alpha < 1 \) can be dealt with analogously after observing that all but the first iterated kernels of \( K(t,z)/(t-z)^\alpha \) are uniformly bounded, and therefore the resolvent series without the first term converges uniformly on the triangle \( 0 \leq z \leq t \leq T \) [7].

For \( N = 1, 2, \ldots, (41) \) and (42) implies that \( w_N(t) \) is a solution of

\[
\int_{-1}^{+1} \int_{-1}^{+1} N_{ss}N_{ss} dz (w)z(w)zt,x,x(G)e^{z(gE)t(w)}
\]  

(A.11)

Hence

\[
|w_N(t)| \leq |E_s| + \int_0^t g(z)G(x_s,x_s,t-z)\sum_{n=N+1}^{L} |w_N(z)|dz, \quad 0 \leq t \leq T
\]  

(A.12)

Since, \( g(z)G(x_s,x_s,t-z) \) is a weakly singular kernel of Volterra type with \( 1/2 < \alpha < 1 \), the lemma implies that the sequence \( \{w_N\} \) is uniformly bounded in the interval \( 0 \leq t \leq T \). Similarly, for \( L > N \)

\[
\int_0^t g(z)G(x_s,x_s,t-z)\sum_{n=N+1}^{L} |w_N(z)|dz + \int_0^t g(z)\sum_{n=N+1}^{L} e^{-\lambda_n(t-z)}|w_N(z)|dz
\]  

(A.13)

There is a constant \( M_1 \) such that for any \( 0 \leq z \leq T \) and any \( n > N \)

\[
|g(z) e_n(x_s)w_N(z)| \leq M_1
\]  

(A.14)

The lemma implies that

\[
|w_L(t) - w_N(t)| \leq \int_0^t g(z)\sum_{n=N+1}^{L} e^{-\lambda_n(t-z)}|w_N(z)|dz + \int_0^t g(z)G(x_s,x_s,t-z)|w_L(z) - w_N(z)|dz
\]  

(A.15)

where \( R(t,z) \) is the resolvent kernel for \( g(z)G(x_s,x_s,t-z) \). Hence, \( \{w_N\} \) is a Cauchy sequence in the Banach space of continuous functions in the interval \( 0 \leq t \leq T \). Therefore it converges uniformly to a continuous function \( w(t) \). (A11) implies that \( w(t) \) is a solution of (37). Since
\[ v_N(x,t) = \int_0^t g(z)(E_z - w_N(z))G_N(x, x_s, t-z)dz \]  \hspace{1cm} (A.16)

and \( G_N(x, x_s, t-z) \) converges to \( G(x, x_s, t-z) \) uniformly in \(-\alpha + \epsilon \leq x \leq \ell, 0 \leq t \leq T\), so does \( v_N(x,t) \) with the limit \( v(x,t) \).
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Value</th>
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<tbody>
<tr>
<td>$C_m$</td>
<td>Membrane capacitance</td>
<td>1 $\mu$F/cm$^2$</td>
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<tr>
<td>$R_m$</td>
<td>Membrane resistance</td>
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</tr>
<tr>
<td>$R_i$</td>
<td>Cytoplasmic specific resistance</td>
<td>100 k$\Omega$·cm</td>
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<td>$\alpha$</td>
<td>Radius of the tip bulb</td>
<td>2-96 $\mu$m</td>
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<tr>
<td>$\beta$</td>
<td>Radius of membrane cylinder</td>
<td>2 $\mu$m</td>
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<tr>
<td>$V_{res}$</td>
<td>Equilibrium potential (-65 mV rescaled to)</td>
<td>0 mV</td>
</tr>
<tr>
<td>$\tau_s$</td>
<td>Synaptic time constant</td>
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<tr>
<td>$\bar{g}_\text{syn}$</td>
<td>Maximum synaptic conductance</td>
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<tr>
<td>$E_{syn}$</td>
<td>Synaptic reversal potential</td>
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<tr>
<td>$\tau_m = C_m R_m$</td>
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<td>6.7 ms</td>
</tr>
<tr>
<td>$\lambda = (\beta R_m / (2R_i))^{1/2}$</td>
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<td>$\ell - x_0$</td>
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<tr>
<td>$L = \beta$</td>
<td>Characteristic length</td>
<td>2 $\mu$m</td>
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