Adaptive Series-Parallel Identification of Dynamical Systems with Uncertain Bifurcations and Chaos

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Abstract—Identifying a real-world dynamical system with uncertain transitions among bifurcations and chaos is a first step in applying the celebrated chaos theory to a real-world phenomenon with such transitions. This paper describes a systematic method of performing such a first step using measurement data alone.

The adaptive identifiers used are adaptive neural networks (i.e. NNs with long- and short-term memories), whose effectiveness for adaptive system identification has been reported in recent IJCNNs. Numerical results are reported of applying such NNs to adaptive series-parallel identification of four well-known dynamical systems with uncertain bifurcations and chaos, namely a predator-prey model, Henon system, blood cell population model, and Lorenz system, which are 2-D, 2-D, 1-D and 3-D respectively. For each of these systems, an adaptive neural network trained on stable trajectories tracks adaptively periodic and chaotic trajectories successfully, showing an impressive generalization ability. An accommodative neural network, which has been shown to have adaptive capability, was trained to a lower value of the same mean squared error criterion on the same data for each dynamical system. However, the accommodative neural networks do not generalize as well to the periodic and chaotic trajectories.

I. INTRODUCTION

The chaos theory has been hailed as one of the greatest achievements in natural sciences in the twentieth century. However, a first step to a real-world application of the theory is to find a mathematical description of the physical, biological, societal or economical phenomenon under study. Such phenomenon may undergo uncertain bifurcations and transitions among stable, periodic and chaotic behaviors. If a mathematical description cannot be derived with principles of physics, biology, sociology or economics, it calls for adaptive identification of the underlying dynamical system.

Neural computing is a most promising approach to system identification. However, standard neural networks (NNs) are not suitable for adaptive processing because adjusting all the weights of a standard neural network online to adapt to an uncertain parameter involves much computation, poor local minima to fall into, and long and unstable transients online. To eliminate these difficulties, adaptive neural networks (NNs) were proposed in [1], [2]. An adaptive NN has long- and short-term memories, which are respectively the nonlinear and linear weights of the NN. The former are independent of the uncertain parameter and are determined in an a priori offline training. Only the linear weights are adjusted online to adapt to the uncertain parameter. Since they enter a quadratic error criterion in a linear fashion, the foregoing difficulties are eliminated.

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This paper reports results of applying adaptive neural networks to the series-parallel identification of four well-known dynamical systems with bifurcations and chaos, namely the predator-prey model, Henon system, blood cell population model, and Lorenz system, which are 2-D, 2-D, 1-D and 3-D respectively. For each of these systems, an adaptive neural network trained on stable trajectories tracks adaptively periodic and chaotic trajectories successfully, showing an impressive generalization ability. An accommodative neural network, which has been shown to have adaptive capability, was trained to a lower value of the same mean squared error criterion on the same data for each dynamical system. However, the accommodative neural networks do not generalize as well to the periodic and chaotic trajectories.

II. SERIES-PARALLEL IDENTIFICATION

The general dynamical system under identification in this paper is a discrete-time causal time-invariant dynamical system:

$$y(t, t_1) = f(y^{t-1}, t_{t-1})$$

where

$$y^{t-1} (t_m) := (y(t-1, t_{t-1}), \ldots, y(t-p, t_{t-m}))$$

where $t_i$ denotes an uncertain parameter with values from the compact set $\Theta$, and $y(t, t_i)$ the $n$-dimensional dynamical state at time $t$. The integer $p$ is unknown, but $p < b$ for some known $b < \infty$. It is assumed that for all $t$, $\| y(t, t_i) \| < B$ for some known $B < \infty$. Note that the evolution of the above dynamical system is completely determined by the initial condition $(y(0), \ldots, y(\infty))$ and the parameter $t_i$; and the dynamical system is a deterministic dynamical system.

The parameter $t_i$ is assumed piecewise constant in $t$ and remains at each constant long enough for the system identifier to adapt to. Nevertheless, we note that what constitute the parameter $t_i$ is usually not known and even if it is known, measurements of it are not available. Therefore, it is not assumed that the parameter $t_i$ is known or measurements of it are available.

The only data available for identification of the dynamical system (1) are measurements of $y(t)$. For offline a priori training of a system identifier with weights $w$, the training...
data set is \( S = \bigcup_{m=1}^{M} S_m \), where \( \omega \) denotes the sample index, \( S_m \) denotes both a collection of sample indices \( \omega \) and the corresponding collection of realizations of \( \hat{y} \) indexed by \( \omega \), \( \theta_m \) are exemplary values of the parameter \( \theta \). \( S \) is assumed to be sufficiently representative of the range of the phenomenon relevant to the application. For online adjustment of the system identifier for adaptation, only one realization \( \{ y(k, \theta_k), k = 1, \ldots, t \} \), of the dynamical state is given at time \( t \), where of course, \( \theta_k \) is piece-wise constant but unknown.

Under the foregoing assumptions, in series-parallel system identification of the dynamical system (1), the state \( y(t-1, \theta_k) \) of the dynamical system (1) is the input to the system identifier at time \( t \). The system identifier is to produce an output \( \hat{y}(t) \) at the same time \( t \) to approximate \( y(t, \theta_k) \) as closely as possible for \( t = 1, 2, \ldots \) with respect to some error criterion.

### III. Adaptive MLPs

In [3], [2], it is proven that under mild conditions, a function \( f(x, \theta) \) can be approximated to any accuracy by an adaptive multilayer perceptron (MLP) whose long-term memory is independent of \( \theta \). This allows the long-term memory to be determined offline in an a priori training. Only the short-term memory need to be adjusted online to adapt to the parameter \( \theta \). Since the short-term memory consists of linear weights of the adaptive MLP, adjusting them online can be performed by any fast LMS or RLS algorithm.

An adaptive MLP is trained on a training data set \( S \) to determine its long-term memory for each of the four dynamical systems with respect to the following a priori training criterion:

\[
Q_o (u, v_1, \ldots, v_M) = \sum y(t, \theta_m, \omega) - \hat{f} (y^{-1}(\theta_m, \omega), u, v_m) \right)^2
\]

where

\[
\sum = \sum_{m=1}^{M} \sum_{\omega \in S_m} \sum_{t=1}^{T}
\]

and \( \hat{f} (y^{-1}(\theta_m, \omega), u, v_m) \) denotes the output at time \( t \) of the adaptive MLP with the long-term memory \( u \) and the short-term memory \( v_m \), that has received \( y^{k-1}(\theta_m, \omega) \), \( k = 1, \ldots, t \), one at a time up to \( t \). In the a priori training, this criterion is minimized over all possible values of \( u, v(\theta_1), \ldots, v(\theta_k) \). Of course, techniques such as cross validation should be used to avoid over-fitting the training data. We denote the values of \( u, v_1, \ldots, v_M \) resulting from this a priori training by \( u^*, v_1^*, \ldots, v_M^* \).

For testing the adaptive MLP as a series-parallel identifier, the following error criterion is used:

\[
Q_o (v) = \sum_{k=1}^{t} \left\| y(k, \theta) - \hat{f} (y^{-1}(\theta, \omega), u^*, v) \right\|^2
\]

Notice that \( u^* \) obtained from the offline a priori training is held fixed and only the linear weights \( v \) of the adaptive MLP are adjusted online.

### IV. Predator-prey model

A predator-prey model is a two-dimensional system:

\[
\begin{align*}
x(t+1) &= rx(t)(1-x(t))-y(t) \\
y(t+1) &= xy(t)
\end{align*}
\]

where the parameter \( r \) belongs to \((0, 4]\). There are 3 stationary states, \( x_1 = (0, 0), x_2 = (1-1/r, 0) \), and \( x_3 = (1/r, 1-2/r) \).

For \( r < 1 \), \( x_1 \) is an attractor. For \( r = 1 \), \( x_1 = x_2 \). For \( 1 < r \leq 2 \), \( x_3 \) is a saddle point. For \( r = 2 \), \( x_2 = x_3 \). For \( 2 < r < 3 \), \( x_3 \) is a sink. For \( r = 3 \), Hopf bifurcation appears. The orbits close to \((1/r, 1-2/r)\) for \( r > 3 \) are attracted by a periodic orbit of period 6 [4]. For \( r > 3.4 \), the system is observed to be chaotic.

Three exemplary values \( 1.6, 2.5, 2.8 \) were chosen of \( r \) for a priori training. Two hundred sequences, each being 40 consecutive i/o pairs, were generated each with a constant \( r \) value randomly selected from the 3 exemplary values and an initial dynamical state \((x_0, y_0)\) selected uniformly from \((0.1, 0.5]^2\). These 200 sequences constituted the a priori training data set \( S \). Offline training was performed on an adaptive MLP with 2:10:2 architecture, yielding an RMSE of 1.844629e–3.

An accommodative MLPWIN 2:9:2 architecture was also trained on the same training data set. The priming length for training was selected to be 5 time points. The final RMSE is 1.387622e–4, which we note is better than the RMSE of 1.844629e–3 for the adaptive MLP.

Online testing was performed using the \( r \) values \( 1.5, 3.3, \) and \( 3.6 \), at which the predator-prey model is stable, periodic and chaotic respectively. Every test sequence was 300 time points long consisting of 100 consecutive i/o pairs for each given \( r \) value in the given order. Four hundred such sequences were simulated with initial dynamical states \((x_0, y_0)\) uniformly selected from \((0.1, 0.5]^2\).

The average absolute error of the adaptive MLP over these 400 sequences is shown in Figure 1, that of the accommodative NN is shown in Figure 2, and how the adaptive MLP tracks the true dynamical state in a typical realization at the change point \( t = 200 \) is illustrated in Figure 3, where the dotted line indicates the output of the adaptive MLP and the dashed line indicates the true output of the dynamical system. Note that
V. A HENON SYSTEM

A Henon system is

\[ \begin{align*}
    x(t+1) &= y(t) + 1 - ax^2(t) \\
    y(t+1) &= bx(t)
\end{align*} \]  

(4)

where \( b = 0.3 \). For \( 0 < a < 0.3675 \), it is stable with one equilibrium as a function of \( a \). Transition to periodic trajectories occurs at \( a = 0.3675 \) and period-doubling bifurcations appear for \( 0.3675 < a < 1.08 \). Chaos happen for \( a > 1.08 \).

Three exemplary values 0.2, 0.33, 0.3 were chosen of \( a \) for a priori training. Two hundred sequences, each being 30 consecutive i/o pairs, were generated each with a constant \( a \) value randomly selected from the 3 exemplary values and an initial dynamical state \((x_0, y_0)\) selected uniformly from \((0.1, 0.5)^2\). These 200 sequences constituted the a priori training data set \( S \). Offline training was performed on an adaptive MLP with 2:9:2 architecture, yielding an RMSE of 7.730469e-4. An accommodative MLPWIN 2:9:2 architecture was trained on the same training data set. The priming length for training was selected as 5. The final RMSE is 1.204631e-4.

Online testing was performed using the \( a \) values, 0.25, 0.37, 1.1, at which the Henon system is stable, periodic and chaotic respectively. Every test sequence was 300 time points long consisting of 100 consecutive i/o pairs for each given \( a \) value in the given order. Four hundred such sequences were simulated with initial dynamical states \((x_0, y_0)\) uniformly selected from \((0.1, 0.6)^2\).

The average absolute error of the adaptive MLP over these 400 sequences is shown in Figure 4, that of the accommodative NN is shown in Figure 5, and how the adaptive MLP tracks the true dynamical state in a typical realization at the change point \( t = 200 \) is illustrated in Figure 6, where the dotted line indicates the output of the adaptive MLP and the dashed line indicates the true output of the dynamical system. Note that the adaptive MLP takes about 20 steps to adjust itself to the new value of \( a \). The accommodative MLPWIN is not included because of its poor performance during online testing.

VI. A BLOOD CELL POPULATION MODEL

A blood cell population model is

\[ x(t+1) = (1 - a)x(t) + bx(t)^re^{-sx(t)}. \]  

(5)

where \( r = 8, \ s = 16, \) and \( b = 1.1 \times 10^6 \).

For every \( a > 0 \), there are three equilibrium points: \( 0 = x_1 < x_2 < x_3 \), where \( x_1 \) is always an attractor, \( x_2 \) is always a
Three exemplary values $\beta;\%,\beta;\%\text{H};\%$ were chosen for a priori training. Two hundred sequences, each being 40 consecutive i/o pairs, were generated each with a constant value randomly selected from the 3 exemplary values and an initial dynamical state chosen uniformly from $\beta;\%\text{L};\%,\beta;\%\text{R}$. These 200 sequences constituted the a priori training data set. Offline training was performed on an adaptive MLP with 1:9:1 architecture, yielding an RMSE of $0.0028783\text{e} - 4$. An accommodative MLPWIN 1:8:8:1 architecture was trained on the same training data set. The priming length of data is selected as 5. The final RMSE is $0.0047280\text{e} - 4$.

Online testing was performed using the $\beta$ values, 0.2, 0.3, 0.8, at which the blood cell model is stable, periodic and chaotic respectively. Every test sequence was 300 time points long consisting of 100 consecutive i/o pairs for each given $\beta$ value in the given order. Four hundred such sequences were simulated with initial dynamical states chosen uniformly from $\beta;\%\beta;\%\beta;\%$. The average absolute error of the adaptive MLP over these 400 sequences is shown in Figure 7, that of the accommodative NN is shown in Figure 8, and how the adaptive MLP tracks the true dynamical state in a typical realization at the change point $t = 200$ is illustrated in Figure 9, where the dotted line indicates the output of the adaptive MLP and the dashed line indicates the true output of the dynamical system. Note that the adaptive MLP takes about 20 steps to adjust itself to the new value of $\beta$. The accommodative MLPWIN is not included because of its poor performance during online testing.

VII. A LORENZ MODEL

A discrete-time Lorenz model of atmosphere is

$$x(t + 1) = x(t) + ah(y(t) - x(t))$$
$$y(t + 1) = y(t) + h(-x(t)z(t) - ax(t) - y(t))$$
$$z(t + 1) = z(t) + h(x(t)y(t) - bz(t) - b(a + r))$$

where $h$ is the time length used for discretizing the original continuous-time Lorenz equation. The shorter $h$, the better the approximation. The $h$ selected here is 0.01.

There are three equilibrium points: $x_0 = (0, 0, -a - r)$, $x_1 = (-\sqrt{b(r - 1)}, -\sqrt{b(r - 1)}, -a - 1)$, and $x_2 = (\sqrt{b(r - 1)}, \sqrt{b(r - 1)}, -a - 1)$. For $r < 1$, $x_0$ is an attractor. For $r = 1$, $x_0 = x_1 = x_2$. For $r > 1$, all three equilibrium points appear and are different, $x_0$ being a saddle point.
For $r > r_2 = \frac{a(a + b + 3)}{(a - b - 1)}$, $x_1$ and $x_2$ are unstable. If $a = 10$ and $b = 8/3$ as proposed by Lorenz, $r_2 = 24.74$. For $r > r_2$, the system is chaotic. $x_0$ is a saddle with 2 stable and 1 unstable direction; $x_1$ and $x_2$ are saddle points with 1 stable and 2 unstable directions.

Three exemplary values 12, 15, 21 were chosen for $r$ for an a priori training. Twenty sequences, each being 1300 consecutive i/o pairs, were generated each with a constant $\alpha$ value randomly selected from the 3 exemplary values and an initial dynamical state $(x_0, y_0, z_0)$ selected uniformly from $(0.3, 0.5)^3$. These 1300 sequences constituted the a priori training data set $S$. Offline training was performed on an adaptive MLP with 3:10:10:3 architecture, yielding an RMSE of $4.653670 \times 2$. An accommodative MLPWIN 3:9:3 architecture is trained with an RMSE of $5.366680 \times 3$ on the same training data set. The priming length of data is selected as 5.

Online testing was performed using the $\alpha$ values, 20, 50, 35, at which the Lorenz model is stable, periodic and chaotic respectively. Every test sequence was 3000 time points long consisting of 1000 consecutive i/o pairs for each given $r$ value in the given order. Four hundred such sequences were simulated with initial dynamical states $(x_0, y_0, z_0)$ uniformly selected from $(0.1, 0.6)^3$.

The average absolute error of the adaptive MLP over these 400 sequences is shown in Figure 10, that of the accommodative NN is shown in Figure 11, and how the adaptive MLP tracks the true dynamical state in a typical realization at the change point $t = 200$ is illustrated in Figure 12, where the dotted line indicates the output of the adaptive MLP and the dashed line indicates the true output of the dynamical system. Note that the adaptive MLP takes about 20 steps to adjust itself to the new value of $r$. The accommodative MLPWIN is not included because of its poor performance during online testing.

VIII. CONCLUSION

The approach of using an adaptive multilayer perceptron (MLP) as an adaptive series-parallel identifier of a dynamical system with uncertain bifurcation and chaos is proposed and has been numerically tested. A predator-prey model, Henon system, blood cell population model, and Lorenz model served as the underlying dynamical systems to be identified. The adaptive MLPs were trained on only the stable trajectories simulated of the dynamical system under identification. Yet, all the adaptive MLPs adaptively track periodic and chaotic trajectories online successfully and always converge in about 20 time points.

A research question is “Can we analyze the trained adaptive MLPs to obtain the same conclusions concerning bifurcation and chaos that we already know about the well-known dynamical systems?”

REFERENCES


