Positive and doubly stochastic maps, and 
majorization in Euclidean Jordan algebras

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Abstract
A positive map between Euclidean Jordan algebras is a (symmetric cone) order preserving linear map. We show that the norm of such a map is attained at the unit element, thus obtaining an analog of the operator/matrix theoretic Russo-Dye theorem. A doubly stochastic map between Euclidean Jordan algebras is a positive, unital, and trace preserving map. We relate such maps to Jordan algebra automorphisms and majorization.

Key Words:
Euclidean Jordan algebra, positive map, doubly stochastic map, majorization

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Dedication: This paper is dedicated, with much admiration and appreciation, to Professor Rajendra Bhatia on the occasion of his 65th birthday.
1 Introduction

This paper deals with positive and doubly stochastic maps, and majorization in Euclidean Jordan algebras.

In the first part of the paper, we describe the spectral norm of a positive map between Euclidean Jordan algebras. To elaborate, consider the space $\mathcal{M}^n$ of all $n \times n$ complex matrices and its real subspace $\mathcal{H}^n$ of all $n \times n$ Hermitian matrices. A complex linear map $\Phi : \mathcal{M}^n \rightarrow \mathcal{M}^k$ is said to be positive [5] if $\Phi$ maps positive semidefinite (Hermitian) matrices in $\mathcal{M}^n$ to positive semidefinite matrices in $\mathcal{M}^k$. For such a map, the following result – a consequence of the so called the Russo-Dye Theorem [5] – holds:

$$||\Phi|| = ||\Phi(I)||,$$

where the norms are operator norms. Now, by writing $||A||_\infty$ for the spectral norm/radius of $A \in \mathcal{H}^n$, observing $||A|| = ||A||_\infty$, and restricting $\Phi$ to $\mathcal{H}^n$, the above relation yields

$$||\Phi||_\infty = ||\Phi(I)||_\infty,$$

(1)

where $||\Phi||_\infty$ is the operator norm of $\Phi : (\mathcal{H}^n, ||\cdot||_\infty) \rightarrow (\mathcal{H}^k, ||\cdot||_\infty)$.

In this paper, we show that (1) extends to any positive map between Euclidean Jordan algebras. A Euclidean Jordan algebra is a finite dimensional real inner product space with a compatible ‘Jordan product’. In such an algebra, the cone of squares induces a partial order with respect to which the positivity of a map is defined. Using the spectral decomposition of an element in such an algebra, one defines eigenvalues and the spectral norm. It is in this setting, a generalization of (1) is proved. We remark that unlike $C^*$-algebras which are associative, Euclidean Jordan algebras are non-associative. Along with the positivity result, we prove an analog of the following ‘spread inequality’ stated for any two positive maps $\Phi_1$ and $\Phi_2$ from $\mathcal{M}^n$ to $\mathcal{M}^k$ and any $x \in \mathcal{H}^n$ [6]:

$$||\Phi_1(x) - \Phi_2(x)|| \leq \lambda^+_1(x) - \lambda^-_n(x),$$

(2)

where $\lambda^+_1(x)$ and $\lambda^-_n(x)$ are the largest and smallest eigenvalues of $x \in \mathcal{H}^n$.

The second part of the paper deals with doubly stochastic maps and majorization. Given two vectors $x$ and $y$ in $R^n$, we say that $x$ is majorized by $y$ and write $x \prec y$ [16] if their decreasing rearrangements $x^\downarrow$ and $y^\downarrow$ satisfy

$$\sum_{1}^{k} x^\downarrow_i \leq \sum_{1}^{k} y^\downarrow_i \quad \text{and} \quad \sum_{1}^{n} x^\downarrow_i = \sum_{1}^{n} y^\downarrow_i$$

for all $1 \leq k \leq n$. When this happens, a theorem of Hardy, Littlewood, and Polya ([3], Theorem II.1.10) says that $x = Dy$, where $D$ is a doubly stochastic matrix (which means that $D$ is a nonnegative matrix with row and column sums equal to one). In addition, by a theorem of Birkhoff ([3], Theorem II.2.3), there exist positive numbers $\lambda_i$ and permutation matrices $P_i$ such that

$$\sum_{1}^{N} \lambda_i = 1 \quad \text{and} \quad x = \sum_{1}^{N} \lambda_i P_i y.$$
Now for given Hermitian matrices $X$ and $Y$ in $\mathcal{H}^n$, one says that $X$ is majorized by $Y$ (see [1], page 234) if $\lambda^\downarrow(X) \prec \lambda^\downarrow(Y)$, where $\lambda^\downarrow(X)$ denotes the vector of eigenvalues of $X$ written in the decreasing order, etc. In this case, it is known, (see e.g., [1], Theorem 7.1) $X = \Phi(Y)$ for some linear map $\Phi : \mathcal{H}^n \to \mathcal{H}^n$ (equivalently, $\Phi : \mathcal{M}^n \to \mathcal{M}^n$) that is doubly stochastic (which means that $\Phi$ is positive, unital, and trace preserving, see Section 2 for definitions). Moreover, there exist positive numbers $\lambda_i$ and unitary matrices $U_i$ such that

$$\sum_{i=1}^{N} \lambda_i = 1 \quad \text{and} \quad X = \sum_{i=1}^{N} \lambda_i U_i^* Y U_i.$$  \hspace{1cm} (4)

With the observations that $\mathbb{R}^n$ and $\mathcal{H}^n$ are two instances of Euclidean Jordan algebras and permutation matrices on $\mathbb{R}^n$ and maps of the form $X \mapsto U^* X U$ (with unitary $U$) on $\mathcal{H}^n$ are Jordan algebra automorphisms (these are invertible linear maps that preserve the Jordan product), we extend the above two results to Euclidean Jordan algebras. By defining $x \prec y$ to mean $\lambda^\downarrow(x) \prec \lambda^\downarrow(y)$, we show that in any Euclidean Jordan algebra $V$, the convex hull of the (algebra) automorphism group $\text{Aut}(V)$ and the set of all doubly stochastic maps $\text{DS}(V)$ are related by

$$\text{conv} \ (\text{Aut}(V)) y \subseteq \text{DS}(V) y \subseteq \{x \in V : x \prec y\} \quad (y \in V),$$

with equalities when $V$ is a simple Euclidean Jordan algebra. In addition, we show that for the Jordan spin algebra $\mathcal{L}^n$,

$$\text{conv} \ (\text{Aut}(\mathcal{L}^n)) = \text{DS}(\mathcal{L}^n).$$  \hspace{1cm} (5)

An outline of the paper is as follows. In Section 2, we cover some preliminary material. Section 3 deals with a consequence of Hirzebruch’s min-max theorem and the spectral norm. In Section 4, we prove analogs of (1) and (2) for Euclidean Jordan algebras. Examples of doubly stochastic maps on Euclidean Jordan algebras are given in Section 5. Majorization results are covered in Section 6. In Sections 7 and 8, we relate Jordan algebra automorphisms and doubly stochastic maps.

## 2 Preliminaries

Throughout this paper, $V$ and $V'$ denote Euclidean Jordan algebras. Recall that a Euclidean Jordan algebra $V$ is a finite dimensional real vector space that carries an inner product $\langle x, y \rangle$ (with induced norm $||x||$) and a bilinear Jordan product $x \circ y$ satisfying the following properties:

(a) $x \circ y = y \circ x$.

(b) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$.

(c) $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$.

It is known that every Euclidean Jordan algebra has a ‘unit’ element $e$ that satisfies $x \circ e = x$ for all $x$. The monograph [8] is our primary source on Euclidean Jordan algebras; for a short summery relevant to our study, see [10].
Clearly, $R^n$, with the usual inner product and componentwise product, is a Euclidean Jordan algebra. It is well-known that every nonzero Euclidean Jordan algebra is isomorphic to a product of the following simple algebras:

(i) the algebra $S^n$ of all $n \times n$ real symmetric matrices,

(ii) the algebra $H^n$ of $n \times n$ complex Hermitian matrices,

(iii) the algebra $Q^n$ of $n \times n$ quaternion Hermitian matrices,

(iv) the algebra $O^3$ of $3 \times 3$ octonion Hermitian matrices, and

(v) the Jordan spin algebra $L^n$, $n \geq 3$.

In all the matrix algebras above, the Jordan product and inner product are respectively given by

$$X \circ Y = \frac{XY + YX}{2} \quad \text{and} \quad \langle X, Y \rangle = \text{Re} \, tr(XY),$$

where the trace of a matrix is the sum of its diagonal elements. The Jordan spin algebra $L^n$ is defined as follows. For $n > 1$, consider $R^n$ with the usual inner product. Writing any element $x \in R^n$ as

$$x = \begin{bmatrix} x_0 \\ \pi \end{bmatrix}$$

with $x_0 \in R$ and $\pi \in R^{n-1}$, we define the Jordan product:

$$x \circ y = \begin{bmatrix} x_0 \\ \pi \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \eta \end{bmatrix} := \begin{bmatrix} \langle x, y \rangle \\ x_0 \eta + y_0 \pi \end{bmatrix}. \quad (6)$$

Then $L^n$ denotes the Euclidean Jordan algebra $(R^n, \circ, \langle \cdot, \cdot \rangle)$.

In a Euclidean Jordan algebra $V$, we let

$$V_+ = \{x \circ x : x \in V\}$$

denote its symmetric cone. This closed convex cone is self-dual and homogeneous. In particular,

$$x \in V_+ \Leftrightarrow \langle x, y \rangle \geq 0 \ \forall \ y \in V_+. \quad (7)$$

In $V$, we write

$$x \geq y \ (\text{or} \ y \leq x) \ \text{when} \ x - y \in V_+.$$

An element $c$ in $V$ is an idempotent if $c^2 = c$; it is a primitive idempotent if it is nonzero and cannot be written as a sum of two nonzero idempotents. The set of all primitive idempotents in $V$, denoted by $\tau(V)$, is a compact set, see [8], page 78. For each $c \in \tau(V)$, we let

$$\bar{c} := \frac{c}{\langle e, c \rangle} \ (= \frac{c}{\|c\|^2}).$$
A **Jordan frame** in $V$ is a set $\{e_1, e_2, \ldots, e_r\}$, where each $e_i$ is a primitive idempotent, $e_i \circ e_j = 0$ for all $i \neq j$, and $e = e_1 + e_2 + \cdots + e_r$. In $V$, any two Jordan frames will have the same number of objects; this number is the **rank** of $V$.

*Throughout this paper, we let $V$ and $V'$ denote Euclidean Jordan algebras with respective symmetric cones $V_+$ and $V'_+$, and respective unit elements $e$ and $e'$. Ranks of $V$ and $V'$ are denoted by $r$ and $s$ respectively.*

Any element $x \in V$ will have a **spectral decomposition**

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_r e_r,$$

where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame and the real numbers $x_1, x_2, \ldots, x_r$ are (called) the ‘eigenvalues’ of $x$. As $\langle e_i, e_j \rangle = 0$ for all $i \neq j$, we note that

$$x_i := \frac{\langle x, e_i \rangle}{\langle e_i, e_i \rangle} = \frac{\langle x, e_i \rangle}{\langle e, e_i \rangle} = \langle x, e_i \rangle, \quad i = 1, 2, \ldots, r.$$

We define $\lambda(x)$ to be any vector in $R^r$ with components $x_1, x_2, \ldots, x_r$ and $\lambda^k(x)$ be its (unique) decreasing rearrangement. In particular, we write

$$\lambda_1^k(x) := \max\{x_1, x_2, \ldots, x_r\} \quad \text{and} \quad \lambda_r^k(x) := \min\{x_1, x_2, \ldots, x_r\}.$$

The **trace** and the **spectral norm** of $x$ are respectively defined by

$$\text{tr}(x) := \sum_{1}^{r} x_i \quad \text{and} \quad ||x||_\infty := \max_{1 \leq i \leq r} |x_i|.$$

Then

$$\langle x, y \rangle_{\text{tr}} := \text{tr}(x \circ y)$$

defines the **canonical inner product** on $V$ which is also compatible with the Jordan product on $V$. Various concepts/results/decompositions remain the same when the given inner product is replaced by the canonical inner product; in particular, for an element, the spectral decomposition, eigenvalues, and trace remain the same. We note that under the canonical inner product, the norm of any element $x$, via (8), is

$$||x||_2 := \sqrt{\langle x, x \rangle_{\text{tr}}} = \sqrt{\sum_{1}^{r} |x_i|^2}.$$

In particular, in this setting, the norm of a primitive idempotent is one and $\text{tr}(x) = \langle x, e \rangle_{\text{tr}}$. When $V$ is simple, the given inner product is a (positive) multiple of the canonical inner product, see Prop. III.4.1 in [8].

Given Euclidean Jordan algebras $V$ and $V'$, we let $\mathcal{B}(V, V')$ denote the set of all (continuous) linear maps from $V$ to $V'$. Note that $\mathcal{B}(V, V')$ is a (finite dimensional) normed linear space with respect to the operator norm. We recall the following.

An **invertible** linear transformation $\Gamma \in \mathcal{B}(V, V)$ is
(i) a (Jordan) algebra automorphism if
\[ \Gamma(x \circ y) = \Gamma(x) \circ \Gamma(y) \quad \forall \ x, y \in V; \]

(ii) a (symmetric) cone automorphism if
\[ \Gamma(V_+) = V_+; \]

(iii) orthogonal if \( \langle \Gamma(x), \Gamma(y) \rangle = \langle x, y \rangle \) for all \( x, y \in V. \)

We recall that a complex linear map \( \Phi : \mathcal{M}^n \to \mathcal{M}^k \) is doubly stochastic [1] if it is positive, unital (which means, \( \Phi \) maps the identity matrix in \( \mathcal{M}^n \) to identity matrix in \( \mathcal{M}^k \)), and trace preserving (that is, \( tr(\Phi(X)) = tr(X) \) for all \( X \in \mathcal{M}^n \)). We have an analogous definition for Euclidean Jordan algebras. A linear map \( \Phi : V \to V' \) is said to be

(a) positive if \( \Phi(V_+) \subseteq V'_+ \),

(b) unital if \( \Phi(e) = e' \),

(c) trace preserving if \( tr(\Phi(x)) = tr(x) \) for all \( x \in V \), and

(d) doubly stochastic if it is positive, unital, and trace preserving.

Recall that the transpose of a linear map \( \Phi : V \to V' \) is the map \( \Phi^T : V' \to V \) defined by:
\[ \langle \Phi(x), y' \rangle = \langle x, \Phi^T(y') \rangle \quad \text{for all} \quad x \in V \text{ and } y' \in V'. \]

We have the following.

(i) The positivity of \( \Phi : V \to V' \) is equivalent to that of \( \Phi^T : V' \to V \). (This is an easy consequence of (7)).

(ii) When \( V \) and \( V' \) carry canonical inner products, the trace preserving (unital) property of \( \Phi \) is equivalent to the unital (respectively, trace preserving) property of \( \Phi^T \). This can be seen by observing
\[ \langle x, e \rangle = tr(x) = tr(\Phi(x)) = \langle \Phi(x), e' \rangle = \langle x, \Phi^T(e') \rangle. \quad (9) \]

In particular, when \( V \) is simple and \( \Phi \in \mathcal{B}(V, V) \), \( \Phi \) is doubly stochastic if and only if its transpose is doubly stochastic.

We respectively write \( \text{Aut}(V), \text{Aut}(V_+), \text{Orth}(V), \) and \( \text{DS}(V) \) for the set of all algebra automorphisms, cone automorphisms, orthogonal maps, and doubly stochastic maps.

It is easy to see that algebra automorphisms of \( R^n \) are just permutation matrices. As noted in [10], algebra automorphisms of \( \mathcal{H}^n \) (of \( S^n \)) are given by \( \Phi(X) = U^*XU \) for some unitary (respectively, real orthogonal) matrix \( U \) and those of \( \mathcal{L}^n \) are given by matrices of the form
\[ \Psi = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}, \]
for some orthogonal matrix \( D \) on \( R^{n-1} \).
Let $V_1$ and $V_2$ be two Euclidean Jordan algebras. We define the Jordan and inner products on $V_1 \times V_2$ by $(x_1, x_2) \circ (y_1, y_2) = (x_1 \circ y_1, x_2 \circ y_2)$ and $\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$. If $\Phi_i \in \text{Aut}(V_i)$ for $i = 1, 2$, then the map $(\Phi_1, \Phi_2) : (x_1, x_2) \mapsto (\Phi_1(x_1), \Phi_2(x_2))$ is in $\text{Aut}(V_1 \times V_2)$. In the following result (which is perhaps known), we state a converse and sketch a proof.

**Proposition 1** Let $V_1$ and $V_2$ be two non-isomorphic simple Euclidean Jordan algebras with $\Phi \in \text{Aut}(V_1 \times V_2)$. Then $\Phi$ is of the form $(\Phi_1, \Phi_2)$ for some $\Phi_i \in \text{Aut}(V_i)$ for $i = 1, 2$.

**Proof.** We assume without loss of generality that $V_1$ and $V_2$ carry canonical inner products. To simplify the notation, we write $\Phi$ in the (block) form:

$$
\Phi = \begin{bmatrix} A & B \\ C & D \end{bmatrix},
$$

where $A : V_1 \to V_1$, $B : V_2 \to V_1$, $C : V_1 \to V_2$, and $D : V_2 \to V_2$ are linear transformations. We show that $A$ and $D$ are algebra automorphisms, and $B$ and $C$ are zero. Now, for $x_i$ and $y_i$ in $V_i$,

$$
A(x_1 \circ y_1) + B(x_2 \circ y_2) = (Ax_1 + Bx_2) \circ (Ay_1 + By_2),
$$

$$
C(x_1 \circ y_1) + D(x_2 \circ y_2) = (Cx_1 + Dx_2) \circ (Cy_1 + Dy_2).
$$

By choosing appropriate $x_i$ and $y_i$, we see that $A, B, C, D$ are algebra homomorphisms (that is, they preserve Jordan products) and $Ax_1 \circ By_2 = 0$ and $Dy_2 \circ Cx_1 = 0$. As the kernels of these four transformations are ideals and $V_i$s are assumed to be simple (which means that they do not contain any non-trivial ideals, see Page 51, [8]), we see that $A$ and $D$ are either zero or invertible, $B$ and $C$ are either zero or one-to-one. In addition, from $\langle Ax_1, By_2 \rangle = 0$ and $\langle Dy_2, Cx_1 \rangle = 0$, we get $A^T B = 0$ and $D^T C = 0$. The cases $A = 0$, $B = 0$ and $C = 0$ and $D = 0$ are not possible due to the invertibility of $\Phi$. The cases $A \neq 0$, $B = 0$ and $C \neq 0$, $D = 0$ are also not possible. Now suppose $A = 0$, $B \neq 0$ and $D = 0$, $C \neq 0$. Then, as $B : V_2 \to V_1$ and $C : V_1 \to V_2$ are one-to-one, the algebras $V_1$ and $V_2$ have equal dimensions, and $B$ (which is an algebra homomorphism) becomes an algebra isomorphism. Thus, $B$ maps Jordan frames to Jordan frames and (hence) preserves trace and (canonical) inner product. In this case, $V_1$ and $V_2$ are isomorphic, contradicting our assumption. Hence, this situation does not arise. In the (last) case when $A \neq 0$, $B = 0$ and $C = 0$, $D \neq 0$, we have $A \in \text{Aut}(V_1)$ and $D \in \text{Aut}(V_2)$. \qed

3 The spectral norm

**Theorem 1** (Hirzebruch [14]) Let $V$ be a Euclidean Jordan algebra of rank $r$. For any element $x \in V$, the smallest and largest eigenvalues are given, respectively, by

$$
\lambda^\downarrow(r)(x) = \min_{c \in \tau(V)} \langle x, c \rangle \quad \text{and} \quad \lambda^\uparrow(r)(x) = \max_{c \in \tau(V)} \langle x, c \rangle.
$$

The above result is usually stated for simple Euclidean Jordan algebras [14], [18]. To see the general case, we write $V$ as a product of simple algebras: $V = V_1 \times V_2 \times \cdots \times V_k$. Then, any
primitive idempotent in $V$ is of the form $(0, 0, \ldots, 0, e_i, 0, 0 \ldots, 0)$ for some primitive idempotent $e_i$ in $V_i$ ($i = 1, 2, \ldots, k$). Also, for any $x$ in $V$, eigenvalues of $x$ come from the eigenvalues of the components of $x$. With these facts, one easily verifies the above result.

**Corollary 1** Let $x, a \in V$ with $a \geq 0$. Then,

(i) $\max_{c \in \tau(V)} |\langle x, c \rangle| = \max_{c \in \tau(V)} |\langle |x|, c \rangle| = ||x||_\infty$.

(ii) $-a \leq x \leq a \Rightarrow ||x||_\infty \leq ||a||_\infty$.

**Proof.** (i) For any $y \geq 0$ in $V$, by the above theorem, $||y||_\infty = \lambda^1(y) = \max_{c \in \tau(V)} |\langle y, c \rangle|$. Now, consider any $x \in V$ with its spectral decomposition $x = \sum_i x_i e_i$. From $y := |x| = \sum_i |x_i| e_i$, we see that $y \geq 0$ and for any $c \in \tau(V)$, $\langle y, c \rangle \geq 0$, that is, $|\langle x, c \rangle| \leq |\langle y, c \rangle|$. Since $||y||_\infty = ||x||_\infty$,

$$\max_{c \in \tau(V)} |\langle x, c \rangle| \leq \max_{c \in \tau(V)} |\langle y, c \rangle| = ||y||_\infty = ||x||_\infty.$$  

However, $|x_i| = |\langle x, e_i \rangle| \leq \max_{c \in \tau(V)} |\langle x, c \rangle|$ for all $i$, and so, $||x||_\infty \leq \max_{c \in \tau(V)} |\langle x, c \rangle|$. Thus we have (i).

(ii) For any $c \in \tau(V)$, from $-a \leq x \leq a \Rightarrow -\langle a, c \rangle \leq \langle x, c \rangle \leq \langle a, c \rangle$ we get

$$|\langle x, c \rangle| \leq |\langle a, c \rangle|.$$ 

Taking the maximum over $\tau(V)$, using (i), we get $||x||_\infty \leq ||a||_\infty$. 

From Item (i) of the Corollary, we get the triangle inequality

$$||x + y||_\infty \leq ||x||_\infty + ||y||_\infty.$$ 

Thus, $|| \cdot ||_\infty$ is a norm on $V$, a fact that is already known, see Theorem 4.1, [20].

4 The norm of a positive map

For a linear map $\Phi : V \to V'$, we define

$$||\Phi||_\infty := \max_{x \in V, ||x||_\infty \leq 1} ||\Phi(x)||_\infty.$$ 

**Theorem 2** Let $\Phi : V \to V'$ be a positive map. Then

$$||\Phi||_\infty = ||\Phi(e)||_\infty.$$ 

**Proof.** For any $x \in V$, by writing the spectral decomposition $x = \sum_i x_i e_i$ and using the fact that $e = \sum_i e_i$, we get $-||x||_\infty e \leq x \leq ||x||_\infty e$. From the linearity and positivity of $\Phi$, we get $-||x||_\infty \Phi(e) \leq \Phi(x) \leq ||x||_\infty \Phi(e)$. Now, Item (ii) in Corollary 1 yields $||\Phi(x)||_\infty \leq ||x||_\infty ||\Phi(e)||_\infty$. Hence, $||\Phi||_\infty \leq ||\Phi(e)||_\infty$. As $||e||_\infty = 1$, we have $||\Phi||_\infty = ||\Phi(e)||_\infty$. 

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Corollary 2  **Suppose** \( \phi : V \to R \) **is a linear functional on** \( V \). **Then** \( ||\phi||_\infty = \phi(e) \) **if and only if** \( \phi \) **is positive.**

**Proof.** Assume first that \( \phi \) is positive. Define \( \Phi : V \to V \) by \( \Phi(x) = \phi(x)e \). Then \( \Phi \) is positive on \( V \). An application of the above theorem gives the equality \( ||\phi||_\infty = \phi(e) \).

To see the converse, let \( ||\phi||_\infty = \phi(e) \). Without loss of generality, let \( \phi(e) = 1 \). For any \( x \in V \) with eigenvalues \( x_1, x_2, \ldots, x_r \),

\[
|\phi(x) - \lambda| = |\phi(x - \lambda e)| \leq ||\phi||_\infty ||x - \lambda e||_\infty = \max_i |x_i - \lambda|
\]

for all \( \lambda \in R \). This implies that \( \phi(x) \) is between \( \min x_i \) and \( \max x_i \). In particular, taking \( x \geq 0 \) (for which \( \min x_i \geq 0 \)), we see that \( \phi(x) \geq 0 \). \( \square \)

**Theorem 3**  **Suppose** \( \Phi : V \to V' \) **is a linear map with** \( ||\Phi||_\infty = 1 \) **and** \( \Phi(e) = e' \). **Then** \( \Phi \) **is positive.**

**Proof.** Without loss of generality, we assume that \( V' \) carries the canonical inner product. We fix \( c' \in \tau(V') \) and define

\[
\phi : V \to R, \quad \phi(x) = \langle \Phi(x), c' \rangle \quad (x \in V).
\]

Then, \( \phi \) is a linear functional with \( \phi(e) = \langle \Phi(e), c' \rangle = \langle e', c' \rangle = 1 \). (We note that as \( V' \) carries the canonical inner product, the norm of any primitive element is one.) In addition,

\[
|\phi(x)| = |\langle \Phi(x), c' \rangle| \leq ||\Phi||_\infty ||x||_\infty \leq ||x||_\infty,
\]

where we have used Item \((i)\) of Corollary 1 for \( \Phi(x) \) in \( V' \). We see that conditions of the previous corollary hold. Hence, \( \phi(x) \geq 0 \) for all \( x \geq 0 \) in \( V \). This means that when \( x \geq 0 \), \( \langle \Phi(x), c' \rangle \geq 0 \) for all \( c' \in \tau(V') \). This leads to, by considering nonnegative linear combinations of elements of \( \tau(V') \), \( \langle \Phi(x), y' \rangle \geq 0 \) for all \( y' \in V'_+ \). By an application of (7), we get \( \Phi(x) \geq 0 \) for any \( x \geq 0 \). This proves that \( \Phi \) is a positive map. \( \square \)

We illustrate the above results by some examples.

**Example 1.** Let \( b \in V \). Then the **quadratic representation** \( P_b \) **is defined by**

\[
P_b(x) = 2b \circ (b \circ x) - b^2 \circ x.
\]

Its norm is given by

\[
||P_b||_\infty = ||b||^2_\infty.
\]

This follows easily from the observations that \( P_b \) is a positive map (see Prop. III.2.2, [8]) and \( P_b(e) = b^2 \) and \( ||b^2||_\infty = ||b||^2_\infty \).

**Example 2.** Let \( L : V \to V \) be a \( Z \)-transformation [11], that is, \( L \) is linear and satisfies

\[
x \geq 0, y \geq 0, \text{ and } \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0.
\]
If $L$ is positive stable (which means that all eigenvalues of $L$ have positive real parts), then $L^{-1}$ is a positive map ([11], Theorem 6). Hence,

$$||L^{-1}||_{\infty} = ||L^{-1}(e)||_{\infty}. $$

We can further specialize by taking $V = S^n$. Then for any $A \in R^{n \times n}$, the transformations

$$L_A(X) := AX + XA^T \quad \text{and} \quad S_A(X) := X - AXA^T \quad (X \in S^n)$$

(which are respectively, the Lyapunov and Stein transformations) are $Z$-transformations on $S^n$. When they are positive stable (which happens when $A$ is positive stable for $L_A$ and Schur stable for $S_A$) we see that the spectral norms of $L_A^{-1}$ and $S_A^{-1}$ are attained at the Identity matrix in $S^n$. These results are known, see e.g., [4], [5].

**Example 3.** Corresponding to a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in $V$, consider the Peirce decomposition $V = \sum_{i \leq j} V_{ij}$ [8], where $V_{ii} := \{x \in V : x \circ e_i = x\}$ and for $i \neq j$, $V_{ij} := \{x \in V : x \circ e_i = \frac{1}{2} x = x \circ e_j\}$. Thus, for any $x \in V$, we can write

$$x = \sum_{i \leq j} x_{ij}, \quad \text{where} \quad x_{ij} \in V_{ij}. \quad (10)$$

As $V_{ii} = Re_i$ for all $i$, we may write $x_{ii} = x_i e_i$ with $x_i \in R$. Then, the above decomposition reads:

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}. \quad (11)$$

As $V_{ij}$s are mutually orthogonal, $tr(x) = x_1 + x_2 + \cdots + x_r$. Also, when $x \in V_+, x_i \geq 0$ for all $i$.

Let $A = [a_{ij}]$ be any $r \times r$ real symmetric matrix. Given (10), we can define the ‘Schur product’ of $A$ and $x$ in $V$ by

$$A \bullet x := \sum_{i \leq j} a_{ij} x_{ij}. \quad (12)$$

Now suppose that $A$ is positive semidefinite and let $\Phi(x) := A \bullet x$. It has been shown in [13], Proposition 2.2, that in this situation, $x \in V_+ \Rightarrow \Phi(x) \in V_+$. This means that $\Phi$ is a positive map. Then,

$$||\Phi||_\infty = ||\Phi(e)||_\infty = ||a_{11}e_1 + a_{22}e_2 + \cdots + a_{rr}e_r||_\infty = \max_{1 \leq i \leq r} |a_{ii}|.$$

The following result is motivated by Theorem 1 in [6].

**Theorem 4** Let $\Phi_1$ and $\Phi_2$ be two positive unital maps from $V$ to $V'$. Then for any $x \in V$,

$$||\Phi_1(x) - \Phi_2(x)||_\infty \leq \lambda_1^+(x) - \lambda_1^+(x).$$
Proof. Fix $x$ in $V$, and let (for ease of notation), $\alpha = \lambda_1^r(x)$ and $\beta = \lambda_1^l(x)$. Then,

$$\alpha e \leq x \leq \beta e.$$  

Using the positivity and the unital properties of $\Phi_i$ ($i = 1, 2$), we get

$$\alpha e' \leq \Phi_1(x) \leq \beta e' \quad \text{and} \quad -\beta e' \leq -\Phi_2(x) \leq -\alpha e'.$$

Hence,

$$(\alpha - \beta)e' \leq \Phi_1(x) - \Phi_2(x) \leq (\beta - \alpha)e'.$$

By Item (ii) in Corollary 1, we get the required inequality. \qed

5 Doubly stochastic maps

Recall that a linear map $\Phi$ on a Euclidean Jordan algebra $V$ is doubly stochastic if it is positive, unital, and trace preserving. It is easy to see that the set of all such maps, $\text{DS}(V)$, is convex, and (from Theorem 2) compact in $\mathcal{B}(V, V)$.

We now list some examples.

Example 4. Consider a doubly stochastic map $\Phi : \mathcal{M}^n \to \mathcal{M}^n$. By definition, this map takes positive semidefinite matrices in $\mathcal{M}^n$ to positive semidefinite matrices and is unital and trace preserving. As every Hermitian matrix is the difference of two positive semidefinite matrices, we see that the restriction of $\Phi$ to $\mathcal{H}^n$ is a doubly stochastic map on the Euclidean Jordan algebra $\mathcal{H}^n$. Conversely, as

$$\mathcal{M}^n = \{A + iB : A, B \in \mathcal{H}^n\},$$

it follows that every (real linear) doubly stochastic map $\Phi$ on $\mathcal{H}^n$ can be extended to a (complex linear) doubly stochastic map $\tilde{\Phi}$ on $\mathcal{M}^n$ by

$$\tilde{\Phi}(A + iB) = \Phi(A) + i\Phi(B).$$

Example 5. Every Jordan algebra automorphism (on a Euclidean Jordan algebra) is a doubly stochastic map. This is because any Jordan algebra automorphism $\Phi$ satisfies $\Phi(x^2) = \Phi(x)^2$ (implying positivity), $\Phi(e) = e$ (unital property), and maps Jordan frames to Jordan frames (implying the trace preserving property). Thus, $\text{Aut}(V) \subseteq \text{DS}(V)$; Denoting the convex hull of a set $S$ by $\text{conv}(S)$, we see that $\text{conv}(\text{Aut}(V))$ is convex, and

$$\text{conv}(\text{Aut}(V)) \subseteq \text{DS}(V). \quad (13)$$

We shall see later that $\text{Aut}(V)$ is also compact in $\mathcal{B}(V, V)$.

Example 6. Let $V$ be of rank $r$. It is easy to see that the map

$$\Phi(x) := \frac{\text{tr}(x)}{r}e,$$
is doubly stochastic.

**Example 7.** We fix a Jordan frame \( \{e_1, e_2, \ldots, e_r\} \) in \( V \) and consider the Peirce decomposition (11). Then the ‘diagonal map’ \( \text{Diag} : V \to V \) defined by

\[
\text{Diag}(x) = x_1 e_1 + x_2 e_2 + \cdots + x_r e_r \quad (x \in V)
\]  

(14)
is doubly stochastic.

**Example 8.** Consider a Jordan frame \( \{e_1, e_2, \ldots, e_r\} \) in \( V \) and the set up of Example 3. let \( A \) be an \( r \times r \) real symmetric positive semidefinite matrix with every diagonal entry one. Then, the map

\[
\Phi(x) := A \bullet x
\]
is doubly stochastic.

**Example 9.** We fix a nonzero idempotent \( c \neq e \) in \( V \), and let

\[
V(c, \gamma) := \{x \in V : x \circ c = \gamma x\},
\]
where \( \gamma \in \{0, \frac{1}{2}, 1\} \). Then the Peirce (orthogonal) decomposition

\[
V = V(c, 1) + V(c, \frac{1}{2}) + V(c, 0)
\]  

(15)
holds [8]. In particular, for any element \( x \in V \), we have

\[
x = u + v + w \quad \text{with} \quad u \in V(c, 1), v \in V(c, \frac{1}{2}), w \in V(c, 0).
\]  

(16)
Now, consider the ‘pinching map’ \( \Phi \) on \( V \) defined by

\[
\Phi(x) := u + w.
\]  

(17)

*We claim that \( \Phi \) is doubly stochastic.* To see this, let \( \omega = c - (e - c) \). By writing the spectral decomposition \( c = e_1 + e_2 + \cdots + e_k + 0 e_{k+1} + \cdots + 0 e_r, 1 \leq k < r \), we see that

\[
\omega = e_1 + e_2 + \cdots + e_k - (e_{k+1} + \cdots + e_r).
\]
As \( \omega^2 = e \), by Prop. II.4.4 in [8], \( P_\omega \) is an algebra automorphism of \( V \). Using the definition of \( P_\omega \) and properties of the spaces \( V(c, \gamma) \), one verifies that

\[
\frac{x + P_\omega(x)}{2} = u + w.
\]  

(18)
This means that the pinching map is the average of two algebra automorphisms, namely the identity transformation and \( P_\omega \). Thus, it is doubly stochastic.

The following result exhibits a connection between doubly stochastic maps and doubly stochastic matrices.
Theorem 5 Let \( \Phi \in \mathcal{B}(V, V') \) be doubly stochastic. Let \( \{e_1, e_2, \ldots, e_r\} \) be a Jordan frame in \( V \) and \( \{f_1, f_2, \ldots, f_s\} \) be a Jordan frame in \( V' \). Then the matrix \( A = [a_{ij}], a_{ij} := \langle \Phi(e_i), f_j \rangle_{tr} \), is a doubly stochastic matrix.

Proof. We assume without loss of generality that both \( V \) and \( V' \) carry canonical inner products. With respect to these inner products, \( \Phi \) continues to have the doubly stochastic property. In particular, as observed previously in (9), the trace preserving property of \( \Phi \) becomes the unital property of the transpose \( \Phi^T \).

Now, \( \Phi(V_+ \subseteq V'_+ \) together with (7) implies that \( A \) is a nonnegative matrix. From \( \Phi(e) = e' \) and the orthogonality relations of a Jordan frame, we have, for each fixed \( j \),

\[
\sum_{i=1}^r a_{ij} = \langle \Phi(e), f_j \rangle = \langle e', f_j \rangle = \langle f_j, f_j \rangle = 1.
\]

As \( \Phi^T(e') = e \), for each fixed \( i \),

\[
\sum_{j=1}^s a_{ij} = \langle \Phi(e_i), e' \rangle = \langle e_i, \Phi^T(e') \rangle = \langle e_i, e \rangle = \langle e_i, e_i \rangle = 1.
\]

Thus \( A \) is doubly stochastic. \( \square \)

Remarks. Suppose for a given map \( \Phi \in \mathcal{B}(V, V') \), the matrix \( [\langle \Phi(e_i), f_j \rangle_{tr}] \) is doubly stochastic for arbitrary Jordan frames in \( V \) and \( V' \). Then, an easy argument shows that \( \Phi \) is doubly stochastic. We omit the proof.

6 Majorization

Given \( x, y \in V \), we say that \( x \) is majorized by \( y \) and write \( x \prec y \) if the eigenvalue vectors of \( x \) and \( y \) satisfy

\[
\lambda(x) \prec \lambda(y).
\]

Theorem 6 For \( x, y \in V \), consider the following statements:

(a) \( x = \Psi(y) \), where \( \Psi \) is a convex combination of algebra automorphisms of \( V \).

(b) \( x = \Phi(y) \), where \( \Phi \) is doubly stochastic on \( V \).

(c) \( x \prec y \).

Then, (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c). Furthermore, when \( V \) is \( \mathbb{R}^n \) or simple, the reverse implications hold.

Proof. (a) \( \Rightarrow \) (b): This follows from (13).

(b) \( \Rightarrow \) (c): Let \( x = \Phi(y) \), where \( \Phi \) is doubly stochastic. Writing the spectral decompositions \( x = \sum_i x_i e_i \) and \( y = \sum_i y_i f_i \), we see that \( \lambda(x) = A \lambda(y) \), where \( A = [\langle \Phi(f_j), e_i \rangle_{tr}] \), \( \lambda(x) = \).
[x_1, x_2, \ldots, x_r]^T$, etc. As $A$ is a doubly stochastic matrix by Theorem 5, we see that $\lambda(x) < \lambda(y)$ proving (c).

Now suppose (c) holds. When $V$ is $\mathbb{R}^n$, the implication (c) $\Rightarrow$ (a) is covered by (3). Now assume that $V$ is simple. Let $x < y$ with spectral decompositions $x = \sum_i x_i e_i$ and $y = \sum_i y_i f_i$; without loss of generality, let $x_1 \geq x_2 \geq \cdots \geq x_r$ and $y_1 \geq y_2 \geq \cdots \geq y_r$ so that $\lambda^i(x) = [x_1, x_2, \ldots, x_r]^T$ and $\lambda^i(y) = [y_1, y_2, \ldots, y_r]^T$. By definition, $\lambda^i(x) < \lambda^i(y)$. Then there exists positive numbers $\mu_i$ and permutation matrices $P_i$ such that $\sum_i^N \mu_i = 1$ and

$$\lambda^i(x) = \sum_{j=1}^N \mu_j \lambda^i(y).$$

Now, each $P_j$ induces a permutation $\sigma_j$ on $\{1, 2, \ldots, r\}$. Let $(P_j(y^i))_i = y_{\sigma_j(i)}$. Then $x_i = \sum_{j=1}^N \mu_j y_{\sigma_j(i)}$. Hence,

$$x = \sum_{i=1}^r x_i e_i = \sum_{i=1}^r \left( \sum_{j=1}^N \mu_j y_{\sigma_j(i)} \right) e_i = \sum_{j=1}^N \mu_j \left( \sum_{i=1}^r y_{\sigma_j(i)} e_i \right).$$

Now for each $j = 1, 2, \ldots, N$, let $\Psi_j^{-1} \in \text{Aut}(V)$ such that $\Psi_j^{-1}(e_i) = f_{\sigma_j(i)}$ for all $i$. Since $V$ is simple, such an automorphism exists, see [8], Theorem IV.2.5. Then $e_i = \Psi_j(f_{\sigma_j(i)})$ for all $i$. Hence,

$$x = \sum_{j=1}^N \mu_j \left( \sum_{i=1}^r y_{\sigma_j(i)} e_i \right) = \sum_{j=1}^N \mu_j \Psi_j \left( \sum_{i=1}^r y_{\sigma_j(i)} f_{\sigma_j(i)} \right) = \sum_{j=1}^N \mu_j \Psi_j(y)$$

as $y = \sum_{i=1}^r y_i f_i = \sum_{i=1}^r y_{\sigma_j(i)} f_{\sigma_j(i)}$ for every $j$. We thus have (a).

Here are some illustrations of the above theorem.

- From Example 7, $\text{Diag}(x) < x$. This is a generalization of the Schur inequality, see [12].

- When $A$ is an $r \times r$ real symmetric positive semidefinite matrix with every diagonal entry one, by Example 8, $A \bullet x < x$. This yields $\lambda(A \bullet x) < \lambda(x)$, extending a result of Bapat and Sunder, see [2], Corollary 2.

- From Example 9, for the pinching map (17), $u + w < x$. This result is proved in [20] for simple algebras, based on case by case analysis.

**Remarks.** Let $W$ be a finite dimensional real inner product space and $G$ be a closed subgroup of the group of orthogonal (that is, inner product preserving) linear maps on $W$. In several works, see for example, [7], [9], [17], [19], the concept of G-majorization is developed and studied. One says that $x$ is $G$-majorized by $y$ in $W$ if $x \in \text{conv}(G)y$ (that is, $x$ is in the convex hull of the $G$-orbit of $y$). If we take $W$ to be a Euclidean Jordan algebra with the canonical inner product, then $G = \text{Aut}(W) \subseteq \text{Orth}(W)$ (see [8], page 57). In this setting, the implication (a) $\Rightarrow$ (c) in Theorem 6 shows that $G$-majorization is stronger than our eigenvalue-majorization. While they
are the same when the algebra is simple, they could be different in the general case, as our next example shows.

**Example 10.** We recall that $S^2$ is the algebra of all real $2 \times 2$ symmetric matrices, with $X \circ Y = \frac{XY + YX}{2}$ and $\langle X, Y \rangle = tr(XY)$. Let $V = R \times S^2$ with $(\lambda, X) \circ (\mu, Y) := (\lambda \mu, X \circ Y)$ and $\langle (\lambda, X), (\mu, Y) \rangle := \lambda \mu + \langle X, Y \rangle$ for $\lambda, \mu \in R$ and $X, Y \in S^2$. Consider two elements of $V$: $p = (0, A)$ and $q = (1, B)$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$}

Clearly, $\lambda^k(p) = \lambda^k(q)$ and so $p \prec q$. Now, it follows from Proposition 1 that algebra automorphisms of $V$ are given by $(\lambda, X) \mapsto (\lambda, U^T XU)$ for some orthogonal matrix $U$. Thus, $p \neq \Psi(q)$ for any $\Psi$ which is a convex combination of algebra automorphisms of $V$. So in this algebra, the implication $(c) \Rightarrow (a)$ does not hold. We also observe that with $G = \text{Aut}(V)$, $p$ is not $G$-majorized by $q$.

Now, consider the map $\Phi$ defined by

$$\Phi : (\lambda, \begin{bmatrix} x & y \\ y & z \end{bmatrix}) \mapsto (x, \begin{bmatrix} \lambda & 0 \\ 0 & z \end{bmatrix}).$$

It is easy to see that $\Phi$ is doubly stochastic. Also, $p = \Phi(q)$. This means that in this algebra,

$$\text{conv}(\text{Aut}(V)) \neq \text{DS}(V).$$

### 7 Relations between $\text{Aut}(V)$, $\text{Aut}(V_+) \text{ and DS}(V)$

**Theorem 7** The following statements are equivalent:

(a) $\Phi \in \text{Aut}(V)$.

(b) $\Phi \in \text{DS}(V)$ and $||\Phi(x)||_\infty = ||x||_\infty$ for all $x \in V$.

(c) $\Phi \in \text{Aut}(V_+)$ and $||\Phi(x)||_2 = ||x||_2$ for all $x \in V$.

**Proof.** We assume without loss of generality that $V$ carries the canonical inner product.

(a) $\Rightarrow$ (b): Let $\Phi \in \text{Aut}(V)$. Clearly, $\Phi$ is doubly stochastic and maps Jordan frames to Jordan frames. In particular, it preserves eigenvalues and so, $||\Phi(x)||_\infty = ||x||_\infty$ for all $x \in V$. Thus, we have (b).

Now assume that (b) holds. We show the following:

(i) $\Phi(c) \in \tau(V)$ for all $c \in \tau(V)$, where $\tau(V)$ denotes the set of all primitive idempotents in $V$;

(ii) $\Phi$ maps Jordan frames to Jordan frames;

(iii) $\Phi$ preserves eigenvalues;

(iv) $\Phi \in \text{Aut}(V_+)$.
To see (i), let \( c \in \tau(V) \). Then (from the positive property) \( \Phi(c) \in V_+ \) and \( \|\Phi(c)\|_\infty = \|c\|_\infty = 1 \). So, in the spectral decomposition \( \Phi(c) = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_r f_r \), every \( \lambda_i \) is nonnegative and \( \max_i \lambda_i = 1 \). Without loss of generality, let \( \lambda_1 = 1 \) so that \( \Phi(c) = f_1 + \sum_2^r \lambda_i f_i \). Since \( \Phi \) preserves the trace, \( 1 = tr(c) = tr(\Phi(c)) = 1 + \sum_2^r \lambda_i \). Thus, \( \sum_2^r \lambda_i = 0 \) and \( \lambda_i = 0 \) for all \( i \geq 2 \). Therefore, \( \Phi(c) = f_1 \in \tau(V) \).

Now, to see (ii), let \( \{e_1, e_2, \ldots, e_r\} \) be any Jordan frame in \( V \). For any \( i \), put \( \Phi(e_i) = p_i \). Then, \( \Phi(e) = e \) implies that \( p_1 + p_2 + \cdots + p_r = e \) and so, \( p_1 \circ p_1 + p_1 \circ p_2 + \cdots + p_1 \circ p_r = p_1 \circ e = p_1 \).

As \( p_1 \circ p_1 = p_1 \) (from (i)), this leads to

\[
p_1 \circ p_2 + \cdots + p_1 \circ p_r = 0.
\]

Taking the trace, we get \( \sum p_i \circ p_i = 0 \) (recall that \( V \) carries the canonical inner product). As \( p_i \in \tau(V) \), from (7), we get \( p_1 \circ p_i = 0 \) for all \( i \geq 2 \). From this and Prop. 6 in [10], we infer that \( p_1 \circ p_i = 0 \) for all \( i \geq 2 \). In a similar way, we show that \( p_i \circ p_j = 0 \) for all \( i \neq j \). Thus, \( \{p_1, p_2, \ldots, p_r\} \) is a Jordan frame. So, (ii) holds.

Now, from the spectral decomposition \( x = \sum i \ x_i e_i \), we get \( \Phi(x) = \sum i \ x_i \Phi(e_i) \). From (ii), we see that the eigenvalues of \( x \) and \( \Phi(x) \) are the same. Thus, \( \|\Phi(x)\|_2^2 = \sum i x_i^2 = \|x\|_2^2 \), proving the second part of (c) as well as the invertibility of \( \Phi \).

As any \( x \in V_+ \) is a finite nonnegative linear combination of elements of \( \tau(V) \) and \( \Phi \) preserves \( \tau(V) \), we see that \( \Phi(V_+) \subseteq V_+ \). We now show that \( \Phi^{-1} \) also preserves \( V_+ \). Let \( c \in \tau(V) \) and \( p := \Phi^{-1}(c) \). Starting with the spectral decomposition \( p = \sum i \alpha_i e_i \), via (ii), we get the spectral decomposition \( c = \Phi(p) = \sum i \alpha_i \Phi(e_i) \). Thus, \( \alpha_i \)'s are the eigenvalues of \( c \); we may let, without loss of generality, \( \alpha_1 = 1 \) and \( \alpha_i = 0 \) for all \( i \geq 2 \). This means that \( p = e_1 \) and so \( \Phi^{-1} c = e_1 \in \tau(V) \). Once again, by considering nonnegative finite linear combinations of elements of \( \tau(V) \), we get \( \Phi^{-1}(V_+) \subseteq V_+ \).

Combining this with \( \Phi(V_+) \subseteq V_+ \), we get \( \Phi \in \text{Aut}(V_+) \). Thus we have (c).

(c) \( \Rightarrow \) (a): The second part of condition (c) says that \( \Phi \) is an isometry (hence orthogonal) under the canonical inner product. So condition (c) means that \( \Phi \in \text{Aut}(V_+) \cap \text{Orth}(V) \). As \( V \) carries the canonical inner product, \( \Phi \in \text{Aut}(V_+) \cap \text{Orth}(V) = \text{Aut}(V) \), see [8], page 57. Thus, we have (a). \( \square \)

**Theorem 8** Suppose \( V \) is simple. Then

\[
\text{Aut}(V) = \text{Aut}(V_+) \cap \text{DS}(V) = \{\Phi \in \text{Aut}(V_+) : \Phi(e) = e\}.
\]

**Proof.** The inclusions \( \text{Aut}(V) \subseteq \text{Aut}(V_+) \cap \text{DS}(V) \subseteq \{\Phi \in \text{Aut}(V_+) : \Phi(e) = e\} \) always hold. When \( V \) is simple, the reverse inclusions follow from

\[
\{\Phi \in \text{Aut}(V_+) : \Phi(e) = e\} = \text{Aut}(V_+) \cap \text{Orth}(V) = \text{Aut}(V),
\]

see [8], Pages 56 and 57, Propositions III.4.3 and I.4.3. \( \square \)

**Remarks.** It follows from Theorem 7 that \( \text{Aut}(V) \) is compact in \( \mathcal{B}(V,V) \). Thus, \( \text{conv}(\text{Aut}(V)) \) is a compact convex subset of \( \text{DS}(V) \). When \( V \) is simple, the equivalence of (a) and (b) in Theorem...
6 shows that pointwise equality
\[
\text{conv}(\text{Aut}(V)) z = \text{DS}(V) z \quad (z \in V)
\]
holds. Example 10 shows that such an equality may not hold in non-simple algebras. We now ask if a stronger equality \(\text{conv}(\text{Aut}(V)) = \text{DS}(V)\) can hold in simple algebras. The following result and the example shows that the equality holds in the algebra \(L^n\) (which is of rank 2), but may fail in other algebras.

**Theorem 9** Every doubly stochastic map on \(L^n\) is a convex combination of algebra automorphisms, that is, it is of the form
\[
A = \sum_{i=1}^{N} \lambda_i \Psi_i,
\]
where \(\lambda_i\)s are positive, \(\sum_{i=1}^{N} \lambda_i = 1\), and
\[
\Psi_i = \begin{bmatrix}
1 & 0 \\
0 & U_i
\end{bmatrix},
\]
for some orthogonal matrix \(U_i\) on \(R^{n-1}\).

Thus,
\[
\text{conv}(\text{Aut}(L^n)) = \text{DS}(L^n).
\]

**Proof.** We recall that for \(n > 1\), \(L^n = (R^n, \circ, \langle \cdot, \cdot \rangle)\), where \(R^n\) carries the usual inner product and the Jordan product is given by (6). The vector \(e\) in \(R^n\) with one in the first slot and zeros elsewhere is the unit element of \(L^n\). Also, a vector \(x = [x_0, \pi]^T \in R \times R^{n-1}\) belongs to \(L^n_+\) (the symmetric cone of \(L^n\)) if and only if \(x_0 \geq ||\pi||_2\). As observed previously, algebra automorphisms of \(L^n\) have the form of (above) \(\Psi_i\) and so a convex combination of such automorphisms is doubly stochastic. Now consider a doubly stochastic map \(A\) on \(L^n\) written in the matrix form:
\[
A = \begin{bmatrix}
a & b^T \\
c & D
\end{bmatrix},
\]
where \(a \in R, b, c \in R^{n-1}\) and \(D \in R^{(n-1) \times (n-1)}\). From \(Ae = e\), we get \(a = 1\) and \(c = 0\). As \(L^n\) is an algebra where the inner product is a multiple of the trace inner product, the trace preserving property of \(A\) is equivalent to \(A^T e = e\). This implies that \(b = 0\). Thus,
\[
A = \begin{bmatrix}
1 & 0 \\
0 & D
\end{bmatrix}. \tag{19}
\]

We now show that \(D\) is a convex combination of orthogonal matrices. Let \(u\) be a vector of norm one in \(R^{n-1}\). As \(x = [1 \ u]^T \in L^n_+\), we must have \(Ax \in L^n_+\). This implies that \(||Du|| \leq 1\). Hence, \(||D|| \leq 1\). (Thus, every doubly stochastic matrix on \(L^n\) is of the form (19) for some ‘contractive’
matrix $D$.

Now, using the singular value decomposition (for the real matrix $D$), we can write

$$D = U \text{diag}(s_1, s_2, \ldots, s_{n-1}) W,$$

where $U$ and $W$ are (real) orthogonal matrices. As $||D|| \leq 1$, we have $0 \leq s_i \leq 1$ for all $i$. Thus, we can write $\text{diag}(s_1, s_2, \ldots, s_{n-1})$ as a convex combination of diagonal matrices $D_i$, each of the form $\text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})$, with $|\varepsilon_j| = 1$ for all $j$. We see that $D$ is a convex combination of orthogonal matrices of the form $U_i := UD_i W$. Using $U_i$, we define the algebra automorphism $\Psi_i$. It follows that $A$ is a convex combination of these algebra automorphisms.

**Example 11.** Consider the simple algebra $\mathcal{H}^3$ of all $3 \times 3$ complex Hermitian matrices. We show that $\text{conv}(\text{Aut}(\mathcal{H}^3)) \neq \text{DS}(\mathcal{H}^3)$ (20) by using an example of Landau and Streater ([15], page 118) given in connection with the extreme points of the set of all doubly stochastic maps on $\mathcal{M}^3$. Consider the map $\Phi : \mathcal{H}^3 \to \mathcal{H}^3$ defined by

$$\Phi(X) := V_1^* XV_1 + V_2^* XV_2 + V_3^* XV_3 \quad (X \in \mathcal{H}^3),$$

where $V_i = \frac{1}{\sqrt{2}} J_i$,

$$J_1 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}.$$

As each $V_i$ is Hermitian and $V_1^2 + V_2^2 + V_3^2 = I$, we see that $\Phi \in \text{DS}(\mathcal{H}^3)$. We now claim that $\Phi \notin \text{conv}(\text{Aut}(\mathcal{H}^3))$. Assuming the contrary, suppose $\Phi$ is a convex combination of $\Phi_k \in \text{Aut}(\mathcal{H}^3)$, $k = 1, 2, \ldots, N$. As observed in Example 4, all these doubly stochastic maps on $\mathcal{H}^3$ can be extended to doubly stochastic maps on $\mathcal{M}^3$. We see that the extension $\tilde{\Phi}$ given by

$$\tilde{\Phi}(X) := V_1^* XV_1 + V_2^* XV_2 + V_3^* XV_3 \quad (X \in \mathcal{M}^3),$$

is a convex combination of $\tilde{\Phi}_k, k = 1, 2, \ldots, N$. However, it has been noted in [15], based on Choi’s characterization of extreme completely positive doubly stochastic maps, that $\tilde{\Phi}$ is an extreme point of the set of all doubly stochastic maps on $\mathcal{M}^3$. Thus, $\tilde{\Phi}_k = \tilde{\Phi}$ for all $k$. It follows that $\Phi = \Phi_k \in \text{Aut}(\mathcal{H}^3)$. However, considering the matrix $E_i$ ($i = 1, 2$) in $\mathcal{H}^3$ with one in the $(i, i)$ slot and zeros elsewhere, we see that $E_1 \circ E_2 = 0$ while $\Phi(E_1) \circ \Phi(E_2) \neq 0$. Thus, $\Phi$ cannot be an algebra automorphism, yielding a contradiction. Hence, we have (20).

**8 Extreme points of DS(V)**

As DS(V) is compact and convex, by a well-known Minkowski-Carathéodory Theorem, DS(V) is the convex hull of (the set of) its extreme points. While the description of all extreme points of DS(V) seems to be a difficult task, in the result below, we show that when $V$ is simple, Jordan algebra automorphisms are extreme points of DS(V).

We first prove a simple proposition.
Proposition 2 Let $V$ be a simple Euclidean Jordan algebra and $\Phi \in DS(V)$. Then,

$$||\Phi(x)|| \leq ||x|| \quad (x \in V).$$

Proof. Since $V$ is simple, the given inner product on $V$ is a positive multiple of the canonical inner product; we may assume without loss of generality that $V$ carries the canonical inner product and $||x||_2 = ||x||$ for all $x$. Now, let $x \in V$ and $y := \Phi(x)$. As $V$ is simple, the implication $(b) \Rightarrow (a)$ in Theorem 6 shows that $y = \Psi(x)$, where $\Psi$ is a convex combination of algebra automorphisms of $V$.

Now, algebra automorphisms preserve the 2-norm of a vector (cf. Theorem 7), we see (by triangle inequality) that $||\Psi(x)||_2 \leq ||x||_2$. Thus, $||\Phi(x)|| \leq ||x||$. $\square$

In the result below, we let $\text{Ext}(DS(V))$ denote the set of all extreme points of $DS(V)$.

Theorem 10 Suppose $V$ is simple. Then

$$\text{Aut}(V) \subseteq \text{Ext}(DS(V)).$$

Moreover, when $n \geq 3$,

$$\text{Aut}(L^n) = \text{Ext}(DS(L^n)).$$

Proof. We assume that $V$ carries the canonical inner product, so $||x|| = ||x||_2$ for all $x \in V$. Let $\Phi \in \text{Aut}(V)$ and suppose it is written as a convex combination $\Phi = \sum_{k=1}^{N} \lambda_k \Phi_k$ of doubly stochastic maps on $V$. As Jordan algebra automorphisms preserve the 2-norm (thus, in our setting, the inner product norm), we have, by the above proposition,

$$||x|| = ||\Phi(x)|| \leq \sum_{k=1}^{N} \lambda_k ||\Phi_k(x)|| \leq \sum_{k=1}^{N} \lambda_k ||x|| = ||x||$$

for all $x \in V$. As the inner product norm is strictly convex, it follows that for each $x \in V$, and $k$, $\Phi_k(x)$ is a positive multiple of $\Phi(x)$. Fix $k$ and write $\Phi_k(x) = \beta(x) \Phi(x)$, where $\beta(x) > 0$. As $\Phi$ and $\Phi_k$ preserve the trace, we see that $\beta(x) = 1$ whenever $tr(x) \neq 0$. Thus, $\Phi_k(x) = \Phi(x)$ whenever $tr(x) \neq 0$. As the set of all such vectors form a dense set in $V$, by continuity, we see that $\Phi_k(x) = \Phi(x)$ for all $x \in V$. Hence $\Phi_k = \Phi$. As this holds for every $k$, we see that $\Phi$ is an extreme point of $DS(V)$. Now the statement about $L^n$ follows from Theorem 9. $\square$

Concluding Remarks. In this paper, we studied some properties of positive and doubly stochastic maps, and their connection to majorization. A number of questions remain, some of which are listed below.

- In Theorem 6, does the implication $(c) \Rightarrow (b)$ hold for a general $V$? The resolution of this question depends on whether any two Jordan frames can be mapped to each other by doubly stochastic maps (on a general algebra).

- When do we have the equality $\text{conv}(\text{Aut}(V)) = DS(V)$?

- What are the extreme points of $DS(V)$?

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References


