SCHUR IDEALS AND HOMOMORPHISMS
OF THE SEMIDEFINITE CONE *

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Abstract. We consider the semidefinite cone $K_n$ consisting of all $n \times n$ real symmetric positive semidefinite matrices. A set $I$ in $K_n$ is said to be a Schur ideal if it is closed under addition, multiplication by nonnegative scalars, and Schur multiplication by any element of $K_n$. A Schur homomorphism of $K_n$ is a mapping of $K_n$ to itself that preserves addition, (nonnegative) scalar multiplication and Schur products. This paper is concerned with Schur ideals and homomorphisms of $K_n$. We show that in the topology induced by the trace inner product, Schur ideals in $K_n$ need not be closed, all finitely generated Schur ideals are closed, and in $K_2$, a Schur ideal is closed if and only if it is a principal ideal. We also characterize Schur homomorphisms of $K_n$ and, in particular, show that any Schur automorphism of $K_n$ is of the form $\Phi(X) = PX^T$ for some permutation matrix $P$.

Key words. Semidefinite cone, Schur ideals and homomorphisms

1. Introduction. Consider the Euclidean space $\mathbb{R}^n$ whose elements are regarded as column vectors. Let $S^n$ denote the set of all real $n \times n$ symmetric matrices. It is well known that $S^n$ is a finite dimensional real Hilbert space under the trace inner product defined by

$$\langle X, Y \rangle = \text{trace}(XY) = \sum_{i,j} x_{ij}y_{ij},$$

where $X = [x_{ij}]$ and $Y = [y_{ij}]$. The space $S^n$ also carries the Schur product (sometimes called Hadamard product) defined by

$$X \circ Y := [x_{ij}y_{ij}].$$

With respect to the usual addition, (real) scalar multiplication, and the Schur product, $S^n$ becomes a commutative algebra. In this algebra, a nonempty set is an ideal if it is closed under addition, scalar multiplication, and (Schur) multiplication by any element of $S^n$. Also, a (Schur) homomorphism of $S^n$ is a linear transformation on $S^n$ that preserves Schur products. We note that every ideal in this algebra, being a finite dimensional linear subspace of $S^n$, is topologically closed (in the topology induced by the trace inner product) and finitely generated. Also, it is easy to describe ideals and homomorphisms of this algebra, see Sections 3 and 6.

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The aim of this paper is to study Schur ideals and homomorphisms of the semidefinite cone. Recall that a matrix \( A \in S^n \) is said to be positive semidefinite if \( x^T A x \geq 0 \) for all \( x \in \mathbb{R}^n \). The set of all positive semidefinite matrices in \( S^n \) is the semidefinite cone and is denoted by \( K_n \).

It is well known that \( K_n \) is a closed convex cone in \( S^n \); this means that \( K_n \) is topologically closed and \( X, Y \in K_n, 0 \leq \lambda \in \mathbb{R} \Rightarrow X + Y \in K_n, \lambda X \in K_n \). A famous theorem of Schur (see [5], Theorem 7.5.3) says that

\[
X, Y \in K_n \Rightarrow X \circ Y \in K_n.
\]

So \( K_n \) is closed under addition, multiplication by nonnegative scalars, and the Schur product. Thus, following [2], we say that \( K_n \) is a semi-algebra (under these operations). Calling nonempty subsets of \( K_n \) that share these properties as sub-semi-algebras of \( K_n \), we note the following examples:

**Example 1.** Let \( D\mathcal{N} \mathcal{N}_n \) denote the set of all \( n \times n \) doubly nonnegative matrices – these are matrices in \( S^n \) that are both nonnegative and positive semidefinite. Clearly, this set is a sub-semi-algebra of \( K_n \).

**Example 2.** Let \( \mathcal{CP}_n \) denote the set of all completely positive matrices in \( S^n \). These are matrices of the form

\[
\sum_{i=1}^{N} uu^T,
\]

where \( u \in \mathbb{R}^n_+ \) (the nonnegative orthant of \( \mathbb{R}^n \)) and \( N \) is a natural number. Alternatively, these are matrices of the form \( BB^T \), where \( B \) is a nonnegative (rectangular) matrix. Since

\[
 uu^T \circ vv^T = (u \circ v) (u \circ v)^T,
\]

where \( u \circ v \) is the componentwise product of two vectors \( u \) and \( v \), it follows that the Schur product of two completely positive matrices is once again a completely positive matrix. Thus, \( \mathcal{CP}_n \) is a sub-semi-algebra of \( K_n \).

Interestingly enough, these two cones, along with \( K_n \), have become very important in optimization, particularly in conic linear programming (where one minimizes a linear function subject to linear constraints over a cone), see for example, [1] and [4]. Thus, the problem of finding/characterizing such sub-semi-algebras of \( K_n \) and studying their properties become interesting and useful. Motivated by this, in this article, we consider special sub-semi-algebras of \( K_n \) called Schur ideals. These are nonempty subsets of \( K_n \) that are closed under addition, multiplication by nonnegative
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scalars, and Schur multiplication by any element of \( K_n \); So, a nonempty set \( I \subseteq K_n \) is a Schur ideal in \( K_n \) if

\[
X, Y \in I, \ Z \in K_n, \ 0 \leq \lambda \in R \Rightarrow X + Y, \lambda X, X \circ Z \in I.
\]

We also consider related Schur homomorphisms of \( K_n \) which are mappings from \( K_n \) to \( K_n \) that preserve addition, (nonnegative) scalar multiplication, and Schur products.

Now, motivated by the problem of describing Schur ideals and homomorphisms of the semidefinite cone, we raise the following questions (using the term ‘closed’ to mean ‘topologically closed’): (a) Is every Schur ideal in \( K_n \) closed? How about finitely generated ones? (b) Is every closed Schur ideal of \( K_n \) finitely generated? (c) What are the Schur homomorphisms/automorphisms of \( K_n \)?

Answering some of these, we show in this article, that

1. Schur ideals in \( K_n \) need not be closed,
2. Finitely generated Schur ideals are always closed,
3. In \( K_2 \), a Schur ideal is closed if and only if it is a principal ideal, and
4. Any Schur homomorphism of \( K_n \) is, up to permutation, conjugation by a 0/1-matrix with each row containing at most one 1 followed by pinching.

In particular, a Schur automorphism of \( K_n \) (this is a bijection of \( K_n \) that preserves the three operations on \( K_n \)) is of the form

\[
\Phi(X) = PXP^T,
\]

where \( P \) is a permutation matrix.

Here is an outline of our paper. Section 2 covers some background material. Section 3 deals with two elementary results characterizing Schur ideals in \( S^n \) and \( K_n \). In Section 4, we give an example of a non-closed Schur ideal in any \( K_n, n \geq 2 \). In Section 5, we show that all finitely generated Schur ideals in \( K_n \) are closed. Finally, in section 6, we describe Schur homomorphisms/automorphisms on \( K_n \).

2. Preliminaries. In the Euclidean space \( \mathbb{R}^n \), we let \( e_1, e_2, \ldots, e_n \) denote the standard unit vectors (so that \( e_i \) has one in the \( i \)th slot and zeros elsewhere). We let

\[
e = e_1 + e_2 + \cdots + e_n \quad \text{and} \quad E := ee^T.
\]

We also let \( E_{ij} \) be a symmetric matrix with ones in the \((i, j)\) and \((j, i)\) slots and zeros elsewhere; in particular, \( E_{ii} \) will have a 1 in the \((i, i)\) slot and zeros elsewhere. A matrix is a 0/1-matrix if each entry is either 0 or 1.

As mentioned previously, we let \( S^n \) denote the space of all \( n \times n \) real symmetric matrices. \( S^n \) becomes a Hilbert space under the trace product \( \langle X, Y \rangle = \text{trace}(XY) = \sum_{i,j} x_{ij} y_{ij} \), where \( X = [x_{ij}] \) and \( Y = [y_{ij}] \). The corresponding norm (called the
Frobenius norm) is given by $||X|| = \sqrt{\sum_{ij} x_{ij}^2}$. Thus, convergence in this space is componentwise. From now on, we say that a set in $S^n$ is closed if it is topologically closed under the Frobenius norm. We note the following property of the inner product:

$$\langle X \circ Y, Z \rangle = \langle X, Y \circ Z \rangle$$

for all $X, Y$ and $Z$ in $S^n$. Throughout the paper, we let

$$K_n := \{ A \in S^n : x^T Ax \geq 0 \ \forall \ x \in \mathbb{R}^n \}.$$ 

It is well known that $K_n$ is a closed convex cone in $S^n$. In particular, as convergence in $S^n$ is componentwise, every bounded sequence in $K_n$ will have a subsequence that converges to an element of $K_n$.

In $K_n$, we have the Löwner ordering:

$$Y \succeq X \quad \text{or} \quad X \preceq Y \quad \text{when} \quad Y - X \in K_n.$$ 

It is well known (see [5], Corollary 7.5.4) that

$$(2.1) \quad U, V \succeq 0 \Rightarrow \langle U, V \rangle \geq 0.$$ 

Thus,

$$(2.2) \quad 0 \preceq X \preceq Y \Rightarrow \langle X, X \rangle \leq \langle X, Y \rangle \leq \langle Y, Y \rangle \Rightarrow ||X|| \leq ||Y||.$$ 

If $X = [x_{ij}] \in K_n$, then all principal minors of $X$ are nonnegative. In particular, considering $1 \times 1$ and $2 \times 2$ principal submatrices, we have for all $i$ and $j$,

$$(2.3) \quad x_{ii} \geq 0 \quad \text{and} \quad x_{ii} x_{jj} \geq x_{ij}^2.$$ 

Thus, if a diagonal entry of $X$ (belonging to $K_n$) is zero, then all elements in the row/column containing this entry will be zero. Also, if some off-diagonal entry is nonzero, then the corresponding diagonal entry will be nonzero.

Let $\text{diag}(K_n)$ denote the set of all diagonal matrices in $K_n$. The interior of $K_n$ will be denoted by $K_n^\circ$. Note that this interior consists of all positive definite matrices of $S^n$ — these are matrices $A$ in $S^n$ satisfying $x^T Ax > 0$ for all $0 \neq x \in \mathbb{R}^n$.

In this paper, the pinching operation/transformation on $S^n$ is defined as follows: By choosing a partition of $\Delta := \{1, 2, \ldots, n\}$, we write any matrix $X \in S^n$ in the block form $X = [X_{\alpha \beta}]$. Then the pinching operation/transformation (corresponding to this partition) takes $X$ to $Y = [Y_{\alpha \beta}]$, where $Y_{\alpha \beta} = X_{\alpha \beta}$ when $\alpha = \beta$ and 0 otherwise.
(This simply means that $Y$ is obtained from $X$ by setting all the off-diagonal blocks to zero.). We write $Y = X_{\text{pinch}}$.

For any finite set $\{C_1, C_2, \ldots, C_N\}$ in $\mathcal{K}_n$, we let

$$I_{\{C_1, C_2, \ldots, C_N\}} := \left\{ \sum C_i \circ X_i : X_i \in \mathcal{K}_n, i = 1, 2, \ldots, N \right\}$$

denote the Schur ideal generated by $\{C_1, C_2, \ldots, C_N\}$. We say that a Schur ideal of $\mathcal{K}_n$ is a principal ideal if it is generated by a single matrix in $\mathcal{K}_n$.

3. Two elementary results on Schur ideals. We start with a description of Schur ideals in $\mathcal{S}^n$, which is perhaps well known. We provide a proof for completeness. In what follows, let

$$\Delta := \{1, 2, 3, \ldots, n\}.$$ 

We say that a subset $I$ of $\Delta \times \Delta$ is symmetric if $(i, j) \in I \Rightarrow (j, i) \in I$.

**Proposition 3.1.** Given any symmetric subset $I$ of $\Delta \times \Delta$, the set

$$I = \{X = [x_{ij}] \in \mathcal{S}^n : x_{kl} = 0 \text{ for all } (k, l) \in I\}$$

is a Schur ideal of $\mathcal{S}^n$. Conversely, every Schur ideal of $\mathcal{S}^n$ arises this way.

**Proof.** The first part of the proposition is easy to verify. We prove the converse part. Let $I$ be any Schur ideal of $\mathcal{S}^n$. If $I = \{0\}$, we take $I = \Delta \times \Delta$. If $I = \mathcal{S}^n$, we take $I = \emptyset$. Now assume that $\{0\} \neq I \neq \mathcal{S}^n$. Define

$$I := \{(k, l) : x_{kl} = 0 \text{ for all } X = [x_{ij}] \in I\}$$

and let $J$ be the complement of $I$ in $\Delta \times \Delta$. Let

$$\bar{I} = \{X = [x_{ij}] \in \mathcal{S}^n : x_{kl} = 0 \forall (k, l) \in I\}.$$ 

Clearly, $I \subseteq \bar{I}$. Now for any $(k, l) \in J$, there is a $Y \in I$ with $y_{kl} \neq 0$. For this $(k, l)$, $E_{kl} = \frac{1}{y_{kl}} Y \circ E_{kl} \in I$. Therefore, for any $X \in \bar{I}$, $X = [x_{ij}] = \sum_{1 \leq i \leq j \leq n} x_{ij} E_{ij} = \sum_{1 \leq k \leq l \leq n, (k, l) \in J} x_{kl} E_{kl} \in \sum_{j} R E_{ij} \in I$. It follows that $I = \bar{I}$. \hfill \blacksquare

The following gives a necessary and sufficient condition for a set to be a Schur ideal in $\mathcal{K}_n$.

**Proposition 3.2.** Let $I$ be a set in $\mathcal{K}_n$ containing zero. Then $I$ is a Schur ideal in $\mathcal{K}_n$ if and only if

(i) $I$ is a convex cone, and

(ii) For every diagonal matrix $D \in \mathcal{S}^n$ and $X \in I$, it holds that $DXD \in I$. 

Proof. Suppose \(I\) is a Schur ideal in \(K_n\). Then for any \(\lambda \geq 0\) and \(X \in I\), we have \(\lambda X \in I\). From this, it follows that \(I\) is a convex cone. Now take any diagonal matrix \(D = diag(r_1, r_2, \ldots, r_n)\) in \(S^n\). Then, for \(r = [r_1, r_2, \ldots, r_n]^T\), \(rr^T \in K_n\) and hence \(DXD = X \circ rr^T \in I\). Thus (ii) holds. The converse is proved by using the fact that any matrix in \(K_n\) is a finite sum of matrices of the form \(rr^T\).

4. A non-closed Schur ideal of \(K_n\). In this section, we show that in any \(K_n\), \(n \geq 2\), there is a Schur ideal that is not closed. First, we will construct such an ideal in the cone \(K_2\) of all \(2 \times 2\) real symmetric positive semidefinite matrices.

Example 3. The set

\[ E = diag(K_2) \cup K_2^2 \]

is a non-closed ideal in \(K_2\).

To see this, let \(\lambda \geq 0\), \(A \in diag(K_2)\), \(B \in K_2^2\) and \(X \in K_2\). Clearly, \(A + A \in diag(K_2)\), \(K_2 + B \in K_2^2\), \(\lambda(A + B) \in E\), and \(A \circ X \in diag(K_2)\). If \(X\) is diagonal, then \(B \circ X \in K_2^2\). Suppose \(X\) is not diagonal so that

\[ X = \begin{bmatrix} x & y \\ y & z \end{bmatrix}, \]

with \(x > 0\), \(z > 0\) and \(xz - y^2 \geq 0\). Now writing

\[ B = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \]

where \(a > 0\), \(c > 0\) and \(ac - b^2 > 0\), we see that \(ax > 0\), \(cz > 0\) and \((ax)(cz) > (b^2)(y^2)\). This means that \(B \circ X \in K_2^2\). Hence in all cases, \((A + B) \circ X \in E\). Thus, \(E\) is an ideal of \(K_2\).

To see that \(E\) is not closed, we produce a sequence in \(E\) whose limit is not in \(E\):

\[ \begin{bmatrix} 1 & 1 - \frac{1}{k} \\ 1 - \frac{1}{k} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \]

Now, in any \(K_n\), \(n \geq 3\), we define the set of block matrices

\[ \mathcal{F} = \left\{ \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} : X \in E \right\}. \]

This is a Schur ideal in \(K_n\) that is not closed.

5. Finitely generated ideals in \(K_n\). The example constructed in the previous section raises the question of which Schur ideals are closed in \(K_n\). The following result partially answers this question.
Theorem 5.1. Every finitely generated Schur ideal in $\mathcal{K}_n$ is closed.

The proof is based on the following two lemmas.

Lemma 5.2. Let $A \in \mathcal{K}_n$ with $\text{diag}(A) > 0$. If $X_k$ is a sequence in $\mathcal{K}_n$ with $A \circ X_k$ convergent, then $X_k$ is a bounded sequence.

Proof. Let $X_k = \begin{bmatrix} x_{ij}^{(k)} \end{bmatrix}$, $A = [a_{ij}]$ and $A \circ X_k \rightarrow Y$. Then, by componentwise convergence, for all $i, 1 \leq i \leq n$, $a_{ii}x_{ii}^{(k)} \rightarrow y_{ii}$ and so $x_{ii}^{(k)} \rightarrow \frac{y_{ii}}{a_{ii}}$. Now, each $X_k$ is positive semidefinite and so by (2.3),

$$x_{ii}^{(k)}x_{jj}^{(k)} \geq (x_{ij}^{(k)})^2$$

for all $i, j = 1, 2, 3, \ldots, n$. As $x_{ii}^{(k)}$ and $x_{jj}^{(k)}$ are bounded sequences, it follows that $x_{ij}^{(k)}$ is also bounded for all $i, j$. Hence $X_k$ is a bounded sequence. This completes the proof.

Lemma 5.3. Let $X_k$ be a sequence in $\mathcal{K}_n$ and $A \in \mathcal{K}_n$ with $A \circ X_k \rightarrow Z \in \mathcal{K}_n$. Then there is an $X \in \mathcal{K}_n$ such that $A \circ X = Z$.

Proof. If $A = 0$, we can take $X = 0$. So, assume $A \neq 0$ and without loss of generality write $A$ in the block form as $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, where $A_1$ is positive semidefinite and $\text{diag}(A_1) > 0$. Writing each $X_k$ in the block form, we get

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} X_1^{(k)} \\ X_2^{(k)} \end{bmatrix} \rightarrow \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}.$$ 

Thus, $A_1 \circ X_1^{(k)} \rightarrow Z_1$, $Z_2 = 0$, and $Z_3 = 0$. Now, from the previous lemma, $X_1^{(k)}$ is a bounded sequence; we may assume that it converges to $X_1$. Then

$$\begin{bmatrix} A_1 \circ X_1^{(k)} & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} A_1 \circ X_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Z_1 \\ 0 \end{bmatrix}.$$ 

Hence

$$A \circ \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} = Z.$$ 

Since $\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{K}_n$, the result follows.

Proof of the theorem. Let $\mathcal{E}$ be an ideal in $\mathcal{K}_n$ generated by a finite number of matrices. For simplicity, we assume that $\mathcal{E}$ is generated by two, say, $A$ and $B$, so that any element of $\mathcal{E}$ is of the form $A \circ X + B \circ Y$ for some $X, Y \in \mathcal{K}_n$. 


To show that $E$ is closed, let $X_k$ and $Y_k$ be sequence of matrices in $K_n$ such that $A \circ X_k + B \circ Y_k \to Z$. We will show that $Z = A \circ X + B \circ Y$ for some $X,Y \in K_n$.

Now, with respect to the Löwner ordering, $0 \preceq A \circ X_k \preceq A \circ X_k + B \circ Y_k$. From (2.2), we get $\|A \circ X_k\| \leq \|A \circ X_k + B \circ Y_k\|$. Thus, the sequence $A \circ X_k$ is bounded. We may assume (by going through a subsequence, if necessary) that $A \circ X_k \to U$. By Lemma 5.3, we may write $U = A \circ X$ for some $X \in K_n$. Similarly, we may assume that $B \circ Y_k \to V$, where $V = B \circ Y$ for some $Y \in K_n$. It follows that $Z = A \circ X + B \circ Y$ with $X,Y \in K_n$. Thus, $E$ is closed.

**Remarks.** By modifying the above proof, one can show the following: If $I$ and $J$ are two closed ideals in $K_n$, then so is $I + J$. In particular, if $I$ is a closed ideal in $K_n$ and

$$I^\perp := \{ Y \in K_n : \langle Y, X \rangle = 0 \ \forall \ X \in I \},$$

then $I + I^\perp$ is a closed ideal in $K_n$. We note, however, that $I^\perp$ may be just $\{0\}$, as in the case of the ideal consisting of all diagonal matrices in $K_n$.

**Remarks.** While every finitely generated ideal in $K_n$ is closed, we do not know if the converse holds. However, in the result below, we show that the converse does hold for $K_2$.

**Theorem 5.4.** Every closed ideal in $K_2$ is a principal ideal. Conversely, every principal ideal in $K_2$ is closed.

We begin with a lemma.

**Lemma 5.5.** Let $E$ be a closed ideal in $K_2$ that contains a matrix with an off-diagonal entry nonzero. Let $u := \sup \left\{ \frac{\beta}{\sqrt{\alpha \gamma}} : \left[ \begin{array}{cc} \alpha & \beta \\ \beta & \gamma \end{array} \right] \in E \right\}$. Then, $\left[ \begin{array}{cc} 1 & u \\ u & 1 \end{array} \right] \in E$. Moreover,

$$\left[ \begin{array}{cc} a & b \\ b & c \end{array} \right] \in E \Rightarrow \left[ \begin{array}{cc} a & \frac{b}{u} \\ \frac{b}{u} & c \end{array} \right] \in K_2.$$

**Proof.** Let $\left[ \begin{array}{cc} \alpha & \beta \\ \beta & \gamma \end{array} \right] \in E$ with $\beta \neq 0$; we may, if necessary, consider the Schur product of this matrix with the positive semidefinite matrix $\left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right]$ and consider...
\( \beta > 0. \) Then \( \alpha, \beta, \) and \( \gamma \) are positive. Since \( \alpha \gamma \geq \beta^2, \) it follows that \( u \) in the Lemma is well defined. Now consider a sequence \( \left[ \begin{array}{c} \alpha_k \\ \beta_k \\ \gamma_k \end{array} \right] \in E \) such that \( \alpha_k, \beta_k, \gamma_k > 0 \) and \( \frac{\beta_k}{\sqrt{\alpha_k \gamma_k}} \to u \) as \( k \to \infty. \) Since \( \left[ \begin{array}{c} \frac{1}{\alpha_k} \\ \frac{1}{\alpha_k} \\ \frac{1}{\gamma_k} \end{array} \right] \) is positive semidefinite (as all its principal minors are non-negative) and \( E \) is an ideal, it follows that

\[
\left[ \begin{array}{c} \alpha_k \\ \beta_k \\ \gamma_k \end{array} \right] \circ \left[ \begin{array}{c} \frac{1}{\alpha_k} \\ \frac{1}{\alpha_k \gamma_k} \\ \frac{1}{\gamma_k} \end{array} \right] = \left[ \begin{array}{c} 1 \\ \beta_k \sqrt{\alpha_k \gamma_k} \\ \frac{\beta_k}{\sqrt{\alpha_k \gamma_k}} \end{array} \right] \in E.
\]

Since \( E \) is closed, letting \( k \to \infty, \) we have

\[
\left[ \begin{array}{c} 1 \\ u \\ 1 \end{array} \right] \in E.
\]

Now for the second part of the Lemma. The implication is obvious if \( b = 0. \) If \( b \neq 0, \) then \( a, c > 0 \) and \( \left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \in E \) and so (by definition of \( u \)), \( u \geq \frac{|b|}{\sqrt{ac}}. \) This shows, by considering the principal minors, that \( \left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \in K_2. \) This completes the proof.

\[ \text{Proof of the theorem.} \] If \( E \) is a principal ideal, then it is closed by Theorem 5.1. For the converse, let \( E \) be a closed ideal in \( K_2. \) Suppose first that \( E \) contains a matrix with a nonzero off-diagonal entry. Then from the above lemma there exists a nonzero number \( u \) with \( U := \left[ \begin{array}{c} 1 \\ u \\ 1 \end{array} \right] \in E. \) Now, the principal ideal generated by \( U \) is contained in \( E. \) The equality

\[
\left[ \begin{array}{cc} a & b \\ b & c \end{array} \right] = \left[ \begin{array}{c} 1 \\ u \\ 1 \end{array} \right] \circ \left[ \begin{array}{c} a \\ b \\ c \end{array} \right]
\]

shows that every element of \( E \) is in this principal ideal. Hence, \( E \) is a principal ideal generated by \( U. \)

Now suppose that every off-diagonal entry of every matrix in \( E \) is zero. Then \( E \) consists of diagonal matrices. In this case, it is easily seen that \( E \) is generated by one of the following:

\[
\left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right], \quad \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \quad \left[ \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right], \quad \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right].
\]

Thus, we have shown that \( E \) is a principal ideal. \[ \blacksquare \]
6. Schur homomorphisms of $K_n$. In this section, we characterize Schur homomorphisms of $K_n$. Recall that $f : K_n \rightarrow K_n$ is a Schur homomorphism if for all $X, Y \in K_n$ and $\lambda \geq 0$ in $\mathbb{R}$, we have

$$f(X + Y) = f(X) + f(Y), \quad f(\lambda X) = \lambda f(X), \quad \text{and} \quad f(X \circ Y) = f(X) \circ f(Y).$$

These are special cases of Schur multiplicative maps (which preserve the Schur products) studied in [3].

To motivate our results, we consider an $n \times n 0/1$-matrix $Q$ with each row containing at most one 1. Then it is easy to see that the mapping $X \mapsto QXQ^T$ is a Schur homomorphism of $K_n$. The process of pinching (corresponding to a partition of $\Delta$) followed by a permutation continues to retain the Schur homomorphism property. In other words, the mapping $X \mapsto P(QXQ^T)$ is a Schur homomorphism of $K_n$, where $P$ denotes a permutation matrix. In what follows, we show that every Schur homomorphism arises this way.

Let $f$ be a Schur homomorphism on $K_n$. Then $f$ can be extended to $S_n$ in the following way. As $S_n = K_n - K_n$, any element $X$ of $S_n$ can be written as $X = A - B$ with $A, B \in K_n$. Then we define

$$F(X) := f(A) - f(B).$$

Note that $F(X)$ is well defined: If $X = A - B = C - D$ with $A, B, C, D \in K_n$, then $A + D = B + C$ and so $f(A) + f(D) = f(B) + f(C)$ yielding $F(X) = f(A) - f(B) = f(C) - f(D)$. It is easily seen that $F : S^n \rightarrow S^n$ is a Schur homomorphism of $S^n$, that is, $F$ is linear and for all $X, Y \in S^n$, $F(X \circ Y) = F(X) \circ F(Y)$.

In the following result (which is perhaps known), we characterize Schur homomorphisms of $S^n$, such as $F$. First, a definition. Recalling our notation $\Delta = \{1, 2, \ldots, n\}$, we say that a mapping $\phi : \Delta \times \Delta \rightarrow \Delta \times \Delta$ is symmetric if $\phi(i, j) = \phi(j, i)$ for all $(i, j)$; for notational convenience, we sometimes write $\phi(ij)$ in place of $\phi(i, j)$. Corresponding to such a mapping, we define $\Phi : S^n \rightarrow S^n$ by

$$(6.1) \quad \Phi(X) = Y \quad \text{where} \quad y_{ij} = x_{\phi(ij)}.$$

**Theorem 6.1.** Given any symmetric mapping $\phi$ on $\Delta \times \Delta$ and a symmetric $0/1$-matrix $Z$, the mapping $F : S^n \rightarrow S^n$ defined by

$$F(X) = Z \circ \Phi(X) \quad (X \in S^n)$$

is a Schur homomorphism on $S^n$. Conversely, every Schur homomorphism of $S^n$ arises this way.
Proof. The first part in the theorem is easily verified. We prove the converse. Consider a Schur homomorphism \( F \) on \( S^n \). In order to simplify the proof and make it more transparent, we will think of matrices \( X \) and \( Y \) as vectors \( x \) and \( y \) in \( R^d \), where \( d = \frac{n^2 + n}{2} \). Then \( F(X) \) can be regarded as a linear transformation \( f(x) = Ax \) on \( R^d \) (for some square matrix \( A \)) satisfying \( A(x \circ y) = A(x) \circ A(y) \). This equation splits into \( d \) independent equations of the form \( a^T(x \circ y) = (a^T x)(a^T y) \), where \( a \) is any row vector of \( A \). From this we see that \( a \) is a 0/1-vector with at most one 1. Thus, \( A \) is a 0/1-matrix with at most one 1 in each row. Reverting back to \( F \), we get the stated result.

Remarks. The above result is analogous to Theorem 1.1 in [3], proved for Schur multiplicative maps on \( R^{m \times n} \) under a ‘nonsingularity’ assumption, but without any linearity assumptions.

Before we present our general result on the Schur homomorphisms of \( K_n \), we introduce some notation and prove a lemma. For \( 1 \leq m \leq n \), let \( \alpha := \{ 1, 2, \ldots, m \} \). Corresponding to a symmetric mapping \( \psi : \alpha \times \alpha \to \Delta \times \Delta \), we define \( \Psi : S^n \to S^m \) by

\[
\Psi(X) = Y, \quad \text{where} \quad y_{ij} = x_{\psi(ij)}. \tag{6.2}
\]

Lemma 6.2. Consider \( \psi \) and \( \Psi \) defined above. Then \( \Psi(K_n) \subseteq K_m \) if and only if there exists an \( m \times n \) 0/1-matrix \( Q \) such that each row of \( Q \) contains exactly one 1 and \( \Psi(X) = QXQ^T \) for all \( X \in K_n \).

Proof. Since the ‘If’ part is obvious, we prove the ‘Only if’ part. Suppose \( \Psi(K_n) \subseteq K_m \). We prove the following statements:

(i) If \( \psi(i,i) = (k,l) \), then \( k = l \).
(ii) If \( i \neq j \), \( \psi(i,i) = (k,k) \) and \( \psi(j,j) = (l,l) \), then \( \psi(i,j) \in \{(k,l),(l,k)\} \). (This implies that \( y_{ij} = x_{\psi(ij)} = x_{kl} \).)

(i) If possible, let \( \psi(i,i) = (k,l) \) with \( k \neq l \). Then there exists a matrix \( X \in K_n \) such that \( x_{kl} = -1 \). But then, for \( Y = \Psi(X) \), \( y_{ii} = x_{\psi(ii)} = x_{kl} = -1 \). This cannot happen as \( Y \), being positive semidefinite, should have all diagonal entries nonnegative. Hence \( k = l \).

(ii) Let \( i \neq j \), \( \psi(i,i) = (k,k) \) and \( \psi(j,j) = (l,l) \). We consider two cases.

Case 1: \( k = l \). Without loss of generality, let \( k = l = 1 \). Suppose \( \psi(i,j) \neq (1,1) \). Consider \( u = [1, 2, 3, \ldots, n]^T \) and \( X = uu^T \in K_n \). Then every entry in \( X \) other than the \( (1,1) \) is bigger than one. Moreover, \( y_{ii} = y_{jj} = x_{11} = 1 \) and \( y_{ij} = x_{\psi(ij)} > 1 \). As \( Y = \Psi(X) \in K_m \), these values of \( Y \) violate (2.3). Hence, \( \psi(i,j) = (1,1) = (k,l) \).

Case 2: \( k \neq l \). Without loss of generality (using Item (i)), let \( \psi(i,i) = (1,1) \) and
\[\psi(j,j) = (2,2).\] We need to show that \(\psi(i,j) \in \{(1,2), (2,1)\}\). Assuming this does not hold, we consider the following \(2 \times 2\) principal submatrix of \(Y\):

\[
\begin{bmatrix}
y_{ii} & y_{ij} \\
y_{ij} & y_{jj}
\end{bmatrix} = \begin{bmatrix}
x_{\psi(ii)} & x_{\psi(ij)} \\
x_{\psi(ij)} & x_{\psi(jj)}
\end{bmatrix} = \begin{bmatrix}
x_{11} & x_{\psi(ij)} \\
x_{\psi(ij)} & x_{22}
\end{bmatrix}.
\]

Case 2.1: Suppose \(\psi(i,j) = (1,1)\). Then, taking \(u = [2,1,0,\ldots,0]^T\), \(X = uu^T\), we see that the above principal submatrix of \(Y\) violates (2.3). Hence this case cannot arise.

Case 2.2: Suppose \(\psi(i,j) = (2,2)\). Then, taking \(u = [1,2,0,\ldots,0]\), \(X = uu^T\), we see that the above principal submatrix of \(Y\) violates (2.3). Hence this case cannot arise.

Case 2.3: Suppose \(\psi(i,j) \not\in \{(1,1), (2,2), (1,2), (2,1)\}\). Then, taking \(X = uu^T\), where \(u = [1,1,2,2,\ldots,2]^T\), we see that even in this case, the above principal submatrix of \(Y\) violates (2.3). We conclude that \(\psi(i,j) \in \{(1,2), (2,1)\}\). Thus, we have (ii).

Having proved Items (i) and (ii), we now define the \(m \times n\) \(0/1\)-matrix \(Q = [q_{ij}]\) as follows: For any pair \((i,k)\) with \(i \in \{1,2,\ldots,m\}\) and \(k \in \{1,2,\ldots,n\}\),

\[q_{ik} = 1 \quad \text{when} \quad \psi(i,i) = (k,k) \quad \text{and} \quad q_{ik} = 0 \quad \text{otherwise}.\]

Then each row of \(Q\) contains exactly one 1 and \(\Psi(X) = QXQ^T\) for all \(X \in S^n\). This completes the proof of the lemma.

We now come to the description of Schur homomorphisms of \(K_n\).

**Theorem 6.3.** Let \(Q\) be an \(n \times n\) \(0/1\) matrix with each row containing at most one 1 and \(P\) be an \(n \times n\) permutation matrix. Given a (specific) pinching operation,

\[f(X) := P(QXQ^T)_{\text{pinch}}P^T \quad (X \in S^n)\]

defines a Schur homomorphism of \(K_n\). Conversely, every Schur homomorphism of \(K_n\) arises this way.

**Proof.** As observed previously, the \(f(X) := P(QXQ^T)_{\text{pinch}}P^T \quad (X \in S^n)\) defines a Schur homomorphism. We prove the converse. Suppose \(f\) is a nonzero Schur homomorphism of \(K_n\). We extend \(f\) to \(F\) on \(S^n\) (see the discussion at the beginning of this section); by Theorem 6.1, there exists a nonzero \(0/1\) symmetric matrix \(Z\) and a symmetric mapping \(\phi\) such that

\[F(X) = Z \circ \Phi(X) \quad \forall \ X \in S^n,\]

where \(\Phi\) is defined by (6.1). Since \(\Phi(E) = E\) (for any mapping \(\phi\)), we see that \(Z = Z \circ E = Z \circ \Phi(E) = F(E) = f(E) \in K_n\). Thus, the \(0/1\) symmetric matrix \(Z\) is also positive semidefinite. Now by Proposition 2 in [6], we see that, up to a
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permutation, Z is of the form

\[
Z = \begin{bmatrix}
Z_1 & 0 & 0 & \cdots & 0 \\
0 & Z_2 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & 0 & Z_k & 0 & 0 \\
0 & \cdots & 0 & 0 & 0
\end{bmatrix},
\]

where \( k \leq n \) and each \( Z_i \) is a matrix of ones. We assume without loss of generality that \( Z \) is of the above form (this induces our permutation \( P \)) and \( F(X) = Z \circ \Phi(X) \) for all \( X \). For each \( i = 1, 2, \ldots, k \), let \( \alpha_i \) be a subset of \( \{1, 2, \ldots, n\} \) such that \( Z_{\alpha_i} = Z_i \).

We let \( m_i := |\alpha_i| \) (the number of elements in \( \alpha_i \)). Clearly, these sets are disjoint.

Now \( (F(X))_{\alpha_i} = Z_i \circ (\Phi(X))_{\alpha_i} = (\Phi(X))_{\alpha_i} \).

We know that for each \( X \in \mathcal{K}_n \), \( F(X) = f(X) \in \mathcal{K}_n \). Thus, for each \( i = 1, 2, \ldots, k \), \( (\Phi(X))_{\alpha_i} \in \mathcal{K}_{m_i} \). Now we define \( \psi_i : \alpha_i \times \alpha_i \to \Delta \times \Delta \) by

\[
\psi_i(k,l) := \phi(k,l).
\]

Then, for the corresponding mapping \( \Psi_i \) we have \( \Psi_i(\mathcal{K}_n) \subseteq \mathcal{K}_{m_i} \). By the previous Lemma, for each \( i = 1, 2, \ldots, k \), there exists an \( m_i \times n \) 0/1-matrix \( Q_i \) (with exactly one 1 in each row) such that \( \Psi_i(X) = Q_iXQ_i^T \). Now, by letting \( Q_{k+1} \) to be the zero matrix, we define the \( n \times n \) 0/1-matrix \( Q \) with blocks \( Q_1, Q_2, \ldots, Q_{k+1} \). An easy computation shows that \( QXQ^T \) has diagonal blocks \( Q_iXQ_i^T \), \( i = 1, 2, \ldots, k+1 \). By setting the off-diagonal blocks to zero (which results in the pinching operation), We see that \( F(X) = (QXQ^T)_{\text{pinch}} \). By taking into consideration the permutation used to rearrange rows and columns of \( Z \), we finally arrive at \( f(X) = P(QXQ^T)_{\text{pinch}} P^T \).

This completes our proof.

The following result characterizes Schur automorphisms of \( \mathcal{K}_n \).

**Corollary 6.4.** Let \( f : \mathcal{K}_n \to \mathcal{K}_n \) be a Schur homomorphism. Then the following are equivalent:

(i) \( f \) is one-to-one (or onto) on \( \mathcal{K}_n \).

(ii) There exists a permutation matrix \( P \) such that \( f(X) = PXP^T \) for all \( X \in \mathcal{K}_n \).

(iii) \( f \) is a Schur automorphism of \( \mathcal{K}_n \).

**Proof.** Let \( f : \mathcal{K}_n \to \mathcal{K}_n \) be a Schur homomorphism. Then, its extension \( F \) to \( \mathcal{S}^n \), from Theorem 6.1, can be written in the form \( F(X) = Z \circ \Phi(X) \) for all \( X \in \mathcal{S}^n \).
(i) ⇒ (ii): Suppose that $f$ is one-to-one. It is easy to see that $F$ is also one-to-one. (When $f$ is onto, that is, $f(K_n) = K_n$, its extension $F$ satisfies $F(K_n) = K_n$. Since $F$ is linear and its range contains an open set, namely, the interior of $K_n$, $F$ is also onto and hence one-to-one.) We now show that $Z = E$ (the matrix of all ones). Suppose, $Z$ has a zero entry, say $z_{ij} = 0$. Then, for any $X$, $\langle F(X), E_{ij} \rangle = (F(X))_{ij} = 0$. Thus, $\langle X, F^T(E_{ij}) \rangle = 0$ for all $X \in S^n$. This implies that $F^T(E_{ij}) = 0$. As $F$ is one-to-one (hence invertible), $F^T$ is also one-to-one. Thus, $E_{ij} = 0$, giving a contradiction. Hence, $Z$ has no zero entries. As $Z$ is a 0/1-matrix, we have $Z = E$. Now, this implies that $F(X) = E \circ \Phi(X) = \Phi(X)$ so that $\Phi(= F)$ is a one-to-one Schur homomorphism on $S^n$ such that $\Phi(K_n) \subseteq K_n$. By the previous lemma, there exists a (square) 0/1-matrix $Q$ with each row containing exactly one 1 such that $F(X) = QXQ^T$ for all $X \in S^n$. As $F$ is one-to-one, it is necessarily invertible. This implies (for example, by looking at $F(X) = I$) that $Q$ is invertible. This means that $Q$ cannot have (two) identical rows, thus showing that $Q$ has exactly one 1 in each column/row. Hence, $Q$ is a permutation matrix.

The implications $(ii) \Rightarrow (iii) \Rightarrow (i)$ are easy to verify or obvious.

Concluding Remarks. Motivated by the problem of describing sub-semi-algebras of $K_n$, in this paper, we studied Schur ideals and homomorphisms of the semidefinite cone. While Schur homomorphisms of $K_n$ are completely characterized, the problem of describing all closed Schur ideals of $K_n$ remains open. In particular, it is not known if every closed Schur ideal of $K_n$ is finitely generated.

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REFERENCES