The automorphism group of a completely positive cone and its Lie algebra

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ABSTRACT

Given a closed cone $\mathcal{C}$ in R^n, we consider the corresponding completely positive (convex) cone $\mathcal{K}$ generated by $\{uu^T : u \in \mathcal{C}\}$ in $\mathcal{S}^n$. Under certain conditions on $\mathcal{C}$, we describe the automorphism group of $\mathcal{K}$ and its corresponding Lie algebra in terms of those of $\mathcal{C} \cup -\mathcal{C}$ and/or $\mathcal{C}$. In particular, we show that when $\mathcal{C}$ is a (closed convex) proper cone, the automorphism groups of $\mathcal{C}$ and $\mathcal{K}$ are isomorphic and their corresponding Lie algebras are isomorphic.

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1. Introduction

Given a closed cone $\mathcal{C}$ in R^n that is not necessarily convex, we consider the corresponding *completely positive cone* $\mathcal{K}$ which is the (closed) convex cone generated by $\{uu^T : u \in \mathcal{C}\}$ in the space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices. This paper deals with the automorphism group of $\mathcal{K}$ and its Lie algebra.

To motivate our discussion, let $\mathcal{C} = R^n$. In this case, the corresponding completely positive cone is $\mathcal{S}^n_+$, the cone of all positive semidefinite matrices in $\mathcal{S}^n$. For this cone, the automorphism group $\mathrm{Aut}(\mathcal{S}^n_+)$ (which consists of invertible linear transformations on $\mathcal{S}^n$ mapping $\mathcal{S}^n_+$ onto itself) and its Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathcal{S}^n_+))$ are known (see [11] or [9], Theorem 13, pp. 150 and [4] or [8], Example 1):

- Every element L in $\mathrm{Aut}(\mathcal{S}^n_+)$ is of the form $\widehat{Q}$, where Q is an invertible matrix on R^n and

$$\widehat{Q}(X) := QXQ^T \quad \text{for all} \quad X \in \mathcal{S}^n.$$

- Every element of $\mathrm{Lie}(\mathrm{Aut}(\mathcal{S}^n_+))$ is of the form L_A for some matrix $A \in R^{n\times n}$, where L_A is defined on $\mathcal{S}^n$ by

$$L_A(X) := AX + XA^T.$$

Note that in the above statements, Q (which is invertible) belongs to $Aut(R^n)$, the automorphism group of the cone R^n, and A belongs to $R^{n\times n}$, the Lie algebra of $Aut(R^n)$. This means that the automorphism group of $\mathcal{S}^n_+$ and its Lie algebra can be described by those of the underlying cone, namely, R^n. This raises the question whether an analogous result holds for other closed cones. In this article, we prove the following.

THEOREM 1. *Let $\mathcal{C}$ be a closed cone in R^n, $\widetilde{\mathcal{C}} := \mathcal{C} \cup -\mathcal{C}$, and $\mathcal{K}$ be the completely positive cone of $\mathcal{C}$. Let $Aut(\widetilde{\mathcal{C}})$ and $Aut(\mathcal{K})$ denote, respectively, the automorphism groups of $\widetilde{\mathcal{C}}$ and $\mathcal{K}$ in $\mathcal{S}^n$. Suppose that $\mathcal{C}$ has nonempty interior. Then the following hold:*

(a) The mapping $Q \mapsto \widehat{Q} : Aut(\widetilde{\mathcal{C}}) \to Aut(\mathcal{K})$ is a two-to-one surjective group homomorphism.

(b) The mapping $A \mapsto L_A : Lie(Aut(\widetilde{\mathcal{C}})) \to Lie(Aut(\mathcal{K}))$ is a Lie algebra isomorphism.

THEOREM 2. *Suppose $\mathcal{C}$ is a closed pointed cone in R^n with nonempty interior. Then,*
(i) the mapping $A \mapsto L_A : Lie(Aut(\mathcal{C})) \to Lie(Aut(\mathcal{K}))$ is a Lie algebra isomorphism.
If, in addition, $\mathcal{C} \setminus \{0\}$ is also connected, then
(ii) the mapping $Q \mapsto \widehat{Q} : Aut(\mathcal{C}) \to Aut(\mathcal{K})$ is a group isomorphism.

In particular, conclusions (i) and (ii) hold when $\mathcal{C}$ is a proper cone, that is, $\mathcal{C}$ is a closed convex pointed cone with nonempty interior.

In Theorem 2, Item (i) is a refinement of a recent result of [6] which says that for a proper cone $\mathcal{C}$, the mapping in Item (i) is injective.

2. Preliminaries

Throughout this paper, $(H, \langle \cdot, \cdot \rangle)$ denotes a real finite dimensional inner product space. Let K denote a closed cone in H, that is, K is closed in H and for $x \in K$, $\lambda \geq 0$ in R we have $\lambda x \in K$. For such a cone, its interior is denoted by $int(K)$ and its dual is given by $K^* := \{y \in H : \langle y, x \rangle \geq 0 \; \forall x \in K\}$. Following [2], we say that a closed cone K is

- *pointed* if $K \cap -K = \{0\}$;
- *proper* if K is a closed *convex* pointed cone with nonempty interior;

Given a linear transformation $L : H \to H$ and a closed cone K in H, we say that

- L is *copositive* on K if $\langle L(x), x \rangle \geq 0$ for all $x \in K$;
- L is *Lyapunov-like* on K if $K \ni x \perp y \in K^* \Rightarrow \langle L(x), y \rangle = 0$;
- L is an *automorphism* of K if L is invertible and $L(K) = K$.

We denote the group of all automorphisms of K by $\mathrm{Aut}(K)$; we say that two objects of $\mathrm{Aut}(K)$ are equal if they take identical values on the entire space H. We denote the Lie algebra of $\mathrm{Aut}(K)$ by $Lie(Aut(K))$. Recall that $L \in Lie(Aut(K))$ if there is a differentiable curve $Q(t) : (-\delta, \delta) \to Aut(K)$ such that $Q(0) = Id$ (Identity transformation) and (derivative) $Q'(0) = L$. As $\mathrm{Aut}(K)$ is a matrix group, its Lie algebra is also given by (see [1], Sec. 7.6)

$$Lie(Aut(K)) := \{L \in \mathcal{B}(H, H) : e^{tL} \in Aut(K) \text{ for all } t \in R\},$$

where $\mathcal{B}(H,H)$ denotes the set of all (bounded) linear transformations on H.

By using the results of [12], we may state the following.

Theorem 3. *For any proper cone K,*

$$L \in Lie(Aut(K)) \Leftrightarrow L \text{ is Lyapunov-like on } K.$$

In the space $H = R^n$, vectors are written as column vectors and the usual inner product is written as $\langle x, y\rangle$ or as $x^T y$. We denote the nonnegative orthant by R^n_+. The set of all $n \times n$ real matrices is denoted by $R^{n\times n}$; *throughout, I denotes the identity matrix in $R^{n\times n}$.* A matrix $A \in R^{n\times n}$ is said to be *positive semidefinite* if it is copositive on R^n.

The space $H = \mathcal{S}^n$ consists of all $n \times n$ real symmetric matrices and carries the trace inner product $\langle X, Y\rangle = trace(XY)$, where the trace of a matrix is the sum of all its diagonal elements. We recall that for any two real matrices A and B, $trace(AB) = trace(BA)$; hence when $B = uu^T$ for a column vector u, we have $trace(AB) = u^T Au$. We denote the (symmetric) cone of all positive semidefinite matrices in $\mathcal{S}^n$ by $\mathcal{S}^n_+$.

Given $A \in R^{n\times n}$, the corresponding *Lyapunov* transformation L_A is defined by

$$L_A(X) = AX + XA^T \quad (X \in \mathcal{S}^n).$$

For any invertible matrix Q in $R^{n\times n}$, we define the transformation $\widehat{Q}$ on $\mathcal{S}^n$ by

$$\widehat{Q}(X) := QXQ^T \quad (X \in \mathcal{S}^n).$$

Proposition 4. *The following statements hold:*

(i) For $A, B \in R^{n\times n}$, $L_A = L_B \Rightarrow A = B$.

(ii) For Q and P invertible in $R^{n\times n}$, $\widehat{Q} = \widehat{P} \Rightarrow Q = \pm P$.

Proof. (i) Since $L_A = L_B \Rightarrow L_{A-B} = 0$, it is enough to show that $L_A = 0 \Rightarrow A = 0$. When $L_A = 0$, we have $AX + XA^T = 0$ for any X in $\mathcal{S}^n$, and in particular, for any arbitrary diagonal matrix X. From this, we easily deduce that $A = 0$.
(ii) Since $\widehat{Q} = \widehat{P} \Rightarrow \widehat{P^{-1}Q} = I$, it is enough to show that $\widehat{Q} = I \Rightarrow Q = \pm I$. Now $\widehat{Q} = I \Rightarrow QXQ^T = X$ for all $X \in \mathcal{S}^n$. By taking $X = e_i e_i^T$ and $X = e_i e_j^T + e_j e_i^T$, where $e_1, e_2, \ldots, e_n$ are the standard coordinate vectors in R^n, we deduce that $Q = \pm I$. $\square$

Throughout this paper, $\mathcal{C}$ denotes a *closed cone* in R^n (which is not necessarily convex) and $\widetilde{\mathcal{C}} := \mathcal{C} \cup -\mathcal{C}$. We note that

$$int(C) \neq \emptyset \Leftrightarrow int(\widetilde{\mathcal{C}}) \neq \emptyset.$$

This can be seen, for example, by an application of the Baire category Theorem: If a closed ball in R^n is a union of two closed sets in R^n, then one of the sets must have nonempty interior in R^n.
(When $\mathcal{C}$ is pointed, the sets $\mathcal{C} \setminus \{0\}$ and $-(\mathcal{C} \setminus \{0\})$ are separated in the sense that each is disjoint from the closure of the other. In this case, a connectedness argument can be used instead of the Baire category Theorem.)

Corresponding to $\mathcal{C}$, the *completely positive cone* $\mathcal{K}$ and the *copositive cone* $\mathcal{E}$ in $\mathcal{S}^n$ are defined, respectively, by

$$\mathcal{K} := \Big\{ \sum uu^T : u \in \mathcal{C} \Big\} \tag{1}$$

and

$$\mathcal{E} := \{A \in \mathcal{S}^n : \ A \text{ copositive on } \mathcal{C}\}, \tag{2}$$

where $\sum uu^T$ denotes a finite sum of matrices of the form uu^T. Note that $\mathcal{K}$ is unchanged if $\mathcal{C}$ is replaced by $\widetilde{\mathcal{C}}$. Also, every element in $\mathcal{K}$ is a matrix of the form BB^T with columns of B coming from $\mathcal{C}$ (equivalently, $u \in \widetilde{\mathcal{C}}$). When $\mathcal{C} = R^n_+$, the objects of $\mathcal{K}$ are called *completely positive matrices* [3].

PROPOSITION 5. *Given a closed cone $\mathcal{C}$, the following statements hold:*

(*i*) *$\mathcal{E}$ is a closed convex cone in $\mathcal{S}^n$ and $\mathcal{K} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{E}$.*

(*ii*) *$\mathcal{K}$ is a closed convex cone and $\mathcal{E}$ is the dual of $\mathcal{K}$.*

(*iii*) *If $\mathcal{C}$ has nonempty interior, then $\mathcal{K}$ and $\mathcal{E}$ are proper cones.*

The proof of this proposition is somewhat routine. That $\mathcal{K}$ is closed in $\mathcal{S}^n$ follows from a standard argument via an application of the Carathéodory Theorem for cones [3]. It is easy to verify that $\mathcal{K}$ is always pointed. If $\mathcal{K} - \mathcal{K} \neq \mathcal{S}^n$, then there is some nonzero A in $\mathcal{S}^n$ which is orthogonal to the subspace $\mathcal{K} - \mathcal{K}$; hence $u^T A u = 0$ for all $u \in \mathcal{C}$. When $int(\mathcal{C})$ is nonempty, say, $d \in int(\mathcal{C})$, for any $x \in R^n$ and small $\varepsilon > 0$, we can let $u = d + \varepsilon x \in \mathcal{C}$ to get $(d + \varepsilon x)^T A (d + \varepsilon x) = 0$. This leads to $x^T A x = 0$ for all x. Since $A \in \mathcal{S}^n$, this implies $Ax = 0$ for all x; hence $A = 0$. This shows that when $int(\mathcal{C}) \neq \emptyset$, $\mathcal{K} - \mathcal{K} = \mathcal{S}^n$. As $\mathcal{K}$ is convex, it follows that $int(\mathcal{K}) \neq \emptyset$. Finally, when $\mathcal{K}$ is proper, its dual $\mathcal{E}$ is also proper, see [2].

PROPOSITION 6. *Let $\mathcal{C}$ be a closed cone. Then the following hold:*

(i) For $u \in \mathcal{C}$ and $x \in R^n$, we have $[xx^T = uu^T \Rightarrow x = u$ or $x = -u]$.

(ii) When $\mathcal{C}$ is pointed, for $u, v \in \mathcal{C}$, we have $[vv^T = uu^T \Rightarrow v = u]$.

Proof. (i) Take $u \in \mathcal{C}$ and $x \in R^n$ with $xx^T = uu^T$. As $x_i^2 = u_i^2$ for all i, we may assume that x and u are nonzero. In this case, $xx^Tx = uu^Tx$ and so $x = \lambda\, u$ for some $\lambda \in R$. Then $xx^T = uu^T$ implies that $\lambda^2 = 1$. Thus, $x = u$ or $x = -u$.
(ii) Let $u, v \in \mathcal{C}$ with $vv^T = uu^T$. From (i), $v = u$ or $v = -u$. If $v = -u$, then $v \in \mathcal{C} \cap -\mathcal{C} = \{0\}$ and so $u = -v = 0$. When $v \neq 0$, we must have $v = u$. $\square$

Recall that for a nonzero element $x \in \mathcal{K}$, the ray $\{\lambda\, x : \lambda \geq 0\} \subseteq \mathcal{K}$ is an *extreme ray* of $\mathcal{K}$ if $y, z \in \mathcal{K}$, $x = y + z \Rightarrow y, z \in \{\lambda\, x : \lambda \geq 0\}$.

In what follows, we denote by $Ext(\mathcal{K})$, the set of all nonzero x for which the corresponding ray is an extreme ray of $\mathcal{K}$.

PROPOSITION 7. *Let $\mathcal{C}$ be a closed cone. Then $Ext(\mathcal{K}) = \{uu^T : 0 \neq u \in \mathcal{C}\}$.*

Proof. By the description of $\mathcal{K}$, it follows that $Ext(\mathcal{K}) \subseteq \{uu^T : 0 \neq u \in \mathcal{C}\}$. Now, suppose that $0 \neq u \in \mathcal{C}$ and $uu^T = \sum_1^N u_i u_i^T$, where $0 \neq u_i \in \mathcal{C}$. Then for any $v \in R^n$ with $u^Tv = 0$, we have $\sum_1^N v^T u_i\, u_i^T v = v^T(uu^T)v = 0$. This leads to $u_i^T v = 0$ for all i. Hence, we have the implication $u^Tv = 0 \Rightarrow u_i^T v = 0$ for every i. This shows that for any i, u_i is a multiple of u, that is $u_i u_i^T$ is a nonnegative multiple of uu^T. Thus uu^T is in $Ext(\mathcal{K})$. $\square$

PROPOSITION 8. *Let A be in $\mathcal{S}^n$ such that every 2×2 minor A is zero. Then, rank of A is zero or one.*

This is well known and easy to show: Under the given condition on A, every minor of order 3×3, and by induction, every minor of order $k \times k$ $(3 \leq k \leq n)$ is zero. Hence, rank of A is zero or one.

3. Proofs of Theorems 1 and 2

In this section, we present the proofs of Theorems 1 and 2 and provide some examples. First, we present a preliminary result.

PROPOSITION 9. *Let $\mathcal{C}$ be any closed cone in R^n. Then the following statements hold:*

(a) For each $Q \in Aut(\widetilde{\mathcal{C}})$, we have $\widehat{Q} \in Aut(\mathcal{K})$.

(b) *The mapping* $Q \mapsto \widehat{Q} : Aut(\widetilde{\mathcal{C}}) \to Aut(\mathcal{K})$ *is a group homomorphism.*

(c) *The mapping* $A \mapsto L_A : Lie(Aut(\widetilde{\mathcal{C}})) \to Lie(Aut(\mathcal{K}))$ *is an injective Lie algebra homomorphism.*

Proof. (a) Let $Q \in Aut(\widetilde{\mathcal{C}})$. For any $u \in \widetilde{\mathcal{C}}$, we have $Qu \in \widetilde{\mathcal{C}}$ and so

$$\widehat{Q}(uu^T) = (Qu)(Qu)^T \in \mathcal{K}.$$

This implies that $\widehat{Q}(\mathcal{K}) \subseteq \mathcal{K}$. Since $Q^{-1} \in Aut(\widetilde{\mathcal{C}})$ and $\widehat{Q}^{-1} = \widehat{Q^{-1}}$, we also have $\widehat{Q}^{-1}(\mathcal{K}) \subseteq \mathcal{K}$. Thus $\widehat{Q} \in Aut(\mathcal{K})$.
(b) It is easy to see that the mapping $Q \mapsto \widehat{Q} : Aut(\widetilde{\mathcal{C}}) \to Aut(\mathcal{K})$ is a group homomorphism under multiplication/composition.
(c) Now, let $A \in Lie(Aut(\widetilde{\mathcal{C}}))$ so that there is a differentiable curve $Q(t)$ in $Aut(\widetilde{\mathcal{C}})$ with $Q(0) = I$ (Identity matrix) and $Q'(0) = A$. Then by (b), $L(t) := \widehat{Q(t)}$ is a differentiable curve in $Aut(\mathcal{K})$ with $L(0) = Id$ (Identity transformation). As

$$L'(t)(X) = Q'(t)XQ(t)^T + Q(t)XQ'(t)^T,$$

we see, by putting $t = 0$, that $L'(0) = L_A$. Thus, $L_A \in Lie(Aut(\mathcal{K}))$. That the linear mapping $A \mapsto L_A$ is a Lie algebra homomorphism follows from $L_{[A,B]} = L_{AB-BA} = L_A L_B - L_B L_A = [L_A, L_B]$. Finally, the injectivity comes from Prop. 4. This completes the proof. ☐

Proof of Theorem 1. (a) In view of the above proposition, the mapping $Q \mapsto \widehat{Q} : Aut(\widetilde{\mathcal{C}}) \to Aut(\mathcal{K})$ is a homomorphism. We now show that it is surjective. Let $L \in Aut(\mathcal{K})$. By the Riesz Representation Theorem, there exist matrices $A_{ij} \in \mathcal{S}^n$ such that for any $X \in \mathcal{S}^n$,

$$L(X) = [\langle A_{ij}, X\rangle].$$

Since L is an automorphism of $\mathcal{K}$, it preserves the extreme rays of $\mathcal{K}$: For any nonzero u in $\mathcal{C}$, there is a nonzero $v \in \mathcal{C}$ such that $L(uu^T) = vv^T$. Thus, $u^T A_{ij} u = v_i v_j$ for all i, j and so

$$(u^T A_{ij} u)(u^T A_{kl} u) = (u^T A_{il} u)(u^T A_{kj} u), \tag{3}$$

for all indices i, j, k, l, at least three of which are distinct.
Now, fix $0 \neq x \in R^n$ and let $d \in int(\mathcal{C})$ (which is nonempty by assumption). Then for all small positive ε, $d + \varepsilon x \in int(\mathcal{C})$, hence (3) holds with u replaced by $d + \varepsilon x$. Expanding and comparing terms containing ε^4, we get

$$(x^T A_{ij} x)(x^T A_{kl} x) = (x^T A_{il} x)(x^T A_{kj} x).$$

This means that $L(xx^T) = [x^T A_{ij} x]$ is a matrix with vanishing 2×2 minors. By Prop. 8, $L(xx^T)$ has rank less than or equal to one. As this holds for any nonzero x in R^n, matrices with rank less than or equal to one in $\mathcal{S}^n$ are mapped, under L, to matrices of the same type. By a result of Lim [10] or Waterhouse [13], there exists an invertible matrix $Q \in R^{n \times n}$ and a real number μ such that $L(X) = \mu\, QXQ^T$ for all $X \in \mathcal{S}^n$. Since $L(uu^T) \in \mathcal{S}^n_+$ for any nonzero $u \in \mathcal{C}$, μ cannot be negative. Also, μ cannot be zero, as L is invertible. We may assume that $\mu = 1$. Thus, there exists an invertible matrix Q such that

$$L(X) = QXQ^T \quad (X \in \mathcal{S}^n).$$

Now, let $0 \neq u \in \mathcal{C}$. As L preserves $Ext(\mathcal{K})$, $(Qu)(Qu)^T = L(uu^T) = vv^T$ for some $0 \neq v \in \mathcal{C}$. From Prop. 6, $Qu = v \in \mathcal{C}$ or $Qu = -v \in -\mathcal{C}$. This shows that $Q(\mathcal{C}) \subseteq \mathcal{C} \cup -\mathcal{C}$ and hence

$$Q(\widetilde{\mathcal{C}}) \subseteq \widetilde{\mathcal{C}}.$$

Now, applying this argument to L^{-1} and to the corresponding Q^{-1}, we get $Q^{-1}(\widetilde{\mathcal{C}}) \subseteq \widetilde{\mathcal{C}}$. This means that $Q \in Aut(\widetilde{\mathcal{C}})$. Thus, we have shown that

$$L = \widehat{Q}, \text{ where } Q \in Aut(\widetilde{\mathcal{C}}).$$

It is clear that for any $Q \in Aut(\widetilde{\mathcal{C}})$, $-Q \in Aut(\widetilde{\mathcal{C}})$ and $\widehat{Q} = \widehat{-Q}$. In view of Prop. 4, each element of $Aut(\mathcal{K})$ has exactly two pre-images in $Aut(\widetilde{\mathcal{C}})$. This completes the proof of (a).

(b) In view of the previous proposition, we need only to show that the specified mapping is surjective. Let $L \in Lie(Aut(\mathcal{K}))$. By Lemma 10 given below, there is a differentiable curve $Q(t)$ in $Aut(\widetilde{\mathcal{C}})$ such that $Q(0) = I$ and $e^{tL}(X) = Q(t)XQ(t)^T$ for all $X \in \mathcal{S}^n$. Now, differentiating both sides of $e^{tL}(X) = Q(t)XQ(t)^T$ (for any fixed X) and evaluating the derivatives at $t = 0$, we get

$$L(X) = AX + XA^T \quad (X \in \mathcal{S}^n),$$

where $A = Q'(0)$. By definition, $A \in Lie(Aut(\widetilde{\mathcal{C}}))$. Thus, $L = L_A$ with $A \in Lie(Aut(\widetilde{\mathcal{C}}))$. This completes the proof of (b). $\square$

LEMMA 10. *Suppose that the mapping* $Q \mapsto \widehat{Q} : Aut(\widetilde{\mathcal{C}}) \to Aut(\mathcal{K})$ *is a two-to-one surjective mapping. Let* $\epsilon > 0$ *so that the open balls* $B(I, \varepsilon)$ *and* $B(-I, \varepsilon)$ *around* I *and* $-I$ *respectively, are disjoint in* $R^{n \times n}$. *Then for any* $L \in Lie(Aut(\mathcal{K}))$, *there is a* $\delta > 0$ *and a (unique) differentiable curve* $Q(t) : (-\delta, \delta) \to Aut(\widetilde{\mathcal{C}}) \cap B(I, \frac{1}{2}\varepsilon)$ *such that* $Q(0) = I$ *and*

$$e^{tL} = \widehat{Q(t)} \quad \forall\, t \in (-\delta, \delta).$$

Furthermore, if $\mathcal{C}$ is pointed, we may choose $\varepsilon > 0$ and $\delta > 0$ so that

$$Q(t) : (-\delta, \delta) \to Aut(\mathcal{C}) \cap B(I, \frac{1}{2}\varepsilon).$$

Proof. Let $L \in Lie(Aut(\mathcal{K}))$ so that $e^{tL} \in Aut(\mathcal{K})$ for all t. By our assumption, there exists $Q(t) \in Aut(\widetilde{\mathcal{C}})$ such that $e^{tL}(X) = Q(t)XQ(t)^T$ for all $X \in \mathcal{S}^n$. We see that $e^{tL}(I) = Q(t)Q(t)^T$; this shows that $\{Q(t) : -1 \leq t \leq 1\}$ is a bounded set in $R^{n\times n}$. Now choose $\delta > 0$ so that for $t \in (-\delta, \delta)$, $Q(t) \in B(I, \varepsilon)$ or $Q(t) \in B(-I, \varepsilon)$. (If this is not true, then by using the boundedness of $\{Q(t) : -1 \leq t \leq 1\}$ and taking appropriate limits, we get a Q that is outside these balls satisfying $Id(X) = QXQ^T$ for all $X \in \mathcal{S}^n$. This would contradict Prop. 4.)
Note that if $Q(t) \in B(-I, \varepsilon)$, then $-Q(t) \in B(I, \varepsilon)$. Since the mapping $Q \mapsto \widehat{Q}$ is two-to-one mapping, in $B(I, \varepsilon)$ we have exactly one $Q(t)$ for each t.
Thus, we may assume that for each $t \in (-\delta, \delta)$, there is a unique $Q(t) \in B(I, \frac{1}{2}\varepsilon) \cap Aut(\widetilde{\mathcal{C}})$. We now claim that this $Q(t)$ is continuous in t. Suppose $t_k \to \bar{t}$ in $(-\delta, \delta)$ and (because of boundedness) $Q(t_{k_m}) \to \overline{Q} \neq Q(\bar{t})$. But then

$$\overline{Q}X\overline{Q}^T = \lim e^{t_{k_m}L}(X) = e^{\bar{t}L}(X) = Q(\bar{t})XQ(\bar{t})^T \quad (X \in \mathcal{S}^n)$$

implies that $\overline{Q} = Q(\bar{t})$ by Prop. 4 and uniqueness in $B(I, \varepsilon)$. This contradiction proves continuity. Now, by taking a smaller δ (if necessary), we show that $Q(t)$ is differentiable on $(-\delta, \delta)$. We show this by proving the differentiability of the first column of $Q(t)$ and repeating the argument for other columns. Let e_1 be the vector in R^n with one in the first slot and zeros elsewhere and $E_1 := e_1e_1^T$. Let $Q(t)e_1 = v(t)$ so that

$$e^{tL}(E_1) = v(t)v(t)^T.$$

Let $\alpha(t)$ denote the first component of $v(t)$. Now, for all t near zero, the $(1, 1)$ component of $e^{tL}(E_1)$, namely $e^{tL}(E_1)_{11}$, is close to one and differentiable in t; thus, $\alpha(t)^2 = e^{tL}(E_1)_{11}$ is nonzero and differentiable at all points near zero. As $\alpha(t)$ is continuous, nonzero, and $\alpha(t)^2$ is differentiable, $\alpha(t)$ is also differentiable near zero. Now, $v(t)$ is $\frac{1}{\alpha(t)}$ times the first column of $e^{tL}(E_1)$, hence differentiable at all points near zero. By a similar argument, we see that all columns of $Q(t)$ are differentiable near zero. Thus, $Q(t)$ is differentiable on some $(-\delta, \delta)$.
Now suppose that $\mathcal{C}$ is pointed. The stated conclusion about $Q(t)$ follows once we show that for all small $\varepsilon > 0$,

$$Aut(\widetilde{\mathcal{C}}) \cap B(I, \varepsilon) \subseteq Aut(\mathcal{C}).$$

Assuming this inclusion to be false for every ε, we can find sequences $x_k \in \mathcal{C}$, $Q_k \in Aut(\widetilde{\mathcal{C}})$ such that $Q_k \to I$ and $Q_k(x_k) \notin \mathcal{C}$. We may assume that $||x_k|| = 1$ for all k and let $\lim x_k = x \in \mathcal{C}$. As $Q_k(x_k) \in -\mathcal{C}$, taking limits, we get $I(x) \in -\mathcal{C}$. Thus, $x \in \mathcal{C} \cap -\mathcal{C}$. Since $||x|| = 1$, we reach a contradiction to the pointedness of $\mathcal{C}$. We thus have the inclusion and the proof is complete. □

Proof of Theorem 2. Suppose that $\mathcal{C}$ is pointed and has nonempty interior. To see Item (i), we proceed as in the proof of Theorem 1. As $Aut(\mathcal{C})$ is a subgroup of $Aut(\widetilde{\mathcal{C}})$ and $Lie(Aut(\mathcal{C})) \subseteq Lie(Aut(\widetilde{\mathcal{C}}))$, the mapping $A \mapsto L_A : Lie(Aut(\mathcal{C})) \to Lie(Aut(\mathcal{K}))$ is an injective Lie algebra homomorphism. To show that this map is surjective, let $L \in Lie(Aut(\mathcal{K}))$. Then we have $e^{tL} \in Aut(\mathcal{K})$ for all $t \in R$. By Theorem 1 and Lemma 10, there is a differentiable curve $Q(t)$ in $Aut(\mathcal{C})$ such that $Q(0) = I$ and $e^{tL} = \widehat{Q(t)}$ for all t near zero. By repeating the proof of part (b) in Theorem 1, we verify that $L = L_A$, where, $A = Q'(0)$ now belongs to $Lie(Aut(\mathcal{C}))$. This completes the proof of (i).
Now suppose, additionally, that $\mathcal{C} \setminus \{0\}$ is connected. Since $Aut(\mathcal{C})$ is a subgroup of $Aut(\widetilde{\mathcal{C}})$, the mapping $Q \mapsto \widehat{Q} : Aut(\mathcal{C}) \to Aut(\mathcal{K})$ is a homomorphism. We now show that this map is surjective and injective. Let $L \in Aut(\mathcal{K})$. Since $int(\mathcal{C}) \neq \emptyset$ we can apply Theorem 1 and get a $Q \in Aut(\widetilde{\mathcal{C}})$ such that $L = \widehat{Q}$. Then $Q(\widetilde{\mathcal{C}}) = \widetilde{\mathcal{C}}$ implies that $Q(\mathcal{C}) \subseteq \mathcal{C} \cup -\mathcal{C}$ and by the invertibility of Q,

$$Q(\mathcal{C} \setminus \{0\}) \subseteq (\mathcal{C} \setminus \{0\}) \cup -(\mathcal{C} \setminus \{0\}).$$

Since $\mathcal{C}$ is pointed, the sets $\mathcal{C} \setminus \{0\}$ and $-(\mathcal{C} \setminus \{0\})$ are separated (in the sense that each is disjoint from the closure of the other). By our assumption, $\mathcal{C} \setminus \{0\}$ is connected; hence $Q(\mathcal{C} \setminus \{0\})$ is also connected. It follows that $Q(\mathcal{C} \setminus \{0\}) \subseteq (\mathcal{C} \setminus \{0\})$ or $Q(\mathcal{C} \setminus \{0\}) \subseteq -(\mathcal{C} \setminus \{0\})$ and by taking closures, $Q(\mathcal{C}) \subseteq \mathcal{C}$ or $Q(\mathcal{C}) \subseteq -\mathcal{C}$; As Q and $-Q$ define the same L, without loss of generality, we may assume that $Q(\mathcal{C}) \subseteq \mathcal{C}$. Now, working with L^{-1} and Q^{-1}, we get $Q^{-1}(\mathcal{C}) \subseteq \mathcal{C}$ or $Q^{-1}(\mathcal{C}) \subseteq -\mathcal{C}$. Since $\mathcal{C}$ is pointed and has nonzero elements, we cannot have $Q(\mathcal{C}) \subseteq \mathcal{C}$ and $Q^{-1}(\mathcal{C}) \subseteq -\mathcal{C}$. Thus, $Q^{-1}(\mathcal{C}) \subseteq \mathcal{C}$. Hence, $Q(\mathcal{C}) = \mathcal{C}$, that is, $Q \in Aut(\mathcal{C})$. Thus, we have shown that for each $L \in Aut(\mathcal{K})$, there is a $Q \in Aut(\mathcal{C})$ such that $L = \widehat{Q}$.
Now for the uniqueness: Now let $P \in Aut(\mathcal{C})$ such that $L = \widehat{P}$. Then, by Prop. 4, $P = \pm Q$. If $P = -Q$, then we have the equality $-\mathcal{C} = -Q(\mathcal{C}) = P(\mathcal{C}) = \mathcal{C}$. However, this cannot happen since $\mathcal{C}$ is pointed and has nonzero elements. Hence we must have $P = Q$. This establishes (ii).

Finally, let $\mathcal{C}$ be a proper cone. Then $\mathcal{C}$ is pointed and $\mathcal{C} \setminus \{0\}$ is convex. Also, $int(\mathcal{C}) \neq \emptyset$. Thus, all the conditions of Theorem 2 are satisfied.

Hence we have statements (i) and (ii). This completes the proof. $\square$

Remark. The proof of Theorem 2 given above actually reveals the following: Suppose $\mathcal{C}$ is a closed pointed cone with nonempty interior and $\mathcal{C}\setminus\{0\}$ is connected. Then

$$Aut(\widetilde{\mathcal{C}}) = Aut(\mathcal{C}) \cup -Aut(\mathcal{C}).$$

We now provide some examples to illustrate our results.

Example 1. Let $\mathcal{C} = R^n$ (or the closed upper half-space in R^n). Then Theorem 1 is applicable and we get the results mentioned in the Introduction. $\square$

We note that Theorem 2 is applicable to any self-dual cone $\mathcal{C}$ (that is, $\mathcal{C} = \mathcal{C}^*$), in particular, to symmetric cones in Euclidean Jordan algebras [5].

Example 2. Let $\mathcal{C} = R^n_+$. In this case, $\mathcal{K}$ is the set of all completely positive matrices [3] and $\mathcal{E}$ is the set of all symmetric copositive matrices. It is well known that every automorphism of R^n_+ is a product of a permutation matrix and a diagonal matrix with positive diagonal. In addition, it easily follows from Theorem 3 that $Lie(Aut(R^n_+))$ is the set of all $n \times n$ diagonal matrices. Now, applying Theorem 2, one can describe $Aut(\mathcal{K})$ and its Lie algebra. $\square$

The following examples show that the mappings $Q \mapsto \widehat{Q}$ and $A \mapsto L_A$ in Theorems 1 and 2 need not be surjective without appropriate conditions on $\mathcal{C}$.

Example 3. For $n > 2$, let $\mathcal{L}^n_+$ denote the so-called ice-cream cone (or the second order cone) given by

$$\mathcal{L}^n_+ = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in R^n : t \in R,\, x \in R^{n-1},\, t \geq \|x\| \right\}$$

and let

$$\mathcal{C} := \partial \mathcal{L}^n_+ = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in R^n : t = \|x\| \right\}.$$

Clearly $\mathcal{C}$ is pointed and $\mathcal{C}\setminus\{0\}$ is connected, but $int(\mathcal{C}) = \emptyset$. (It may be instructive to visualize $\mathcal{C}$ in R^3.) Let $J_n = diag(1, -1, \ldots, -1) \in R^{n\times n}$ and $\Gamma(X) := \langle X, J_n \rangle$ for any $X \in \mathcal{S}^n$. Since $\mathcal{K} - \mathcal{K} \subseteq \ker(\Gamma)$, we see that $\mathcal{K} - \mathcal{K} \neq \mathcal{S}^n$.

Let L be an invertible linear transformation on $\mathcal{S}^n$ such that L coincides with the Identity transformation on $\mathcal{K} - \mathcal{K}$, but not on the entire $\mathcal{S}^n$. (For example, writing $\mathcal{S}^n = (\mathcal{K}-\mathcal{K}) \oplus (\mathcal{K}-\mathcal{K})^\perp$, we may define $L(x+y) = x+2y$

for $x \in \mathcal{K}-\mathcal{K}$ and $y \in (\mathcal{K}-\mathcal{K})^{\perp}$.) Then $L \in Aut(\mathcal{K})$. Assume that there is a $Q \in Aut(\widetilde{\mathcal{C}})$ such that $L = \widehat{Q}$, that is, $L(X) = QXQ^T$ for every $X \in \mathcal{S}^n$. Then for all $u \in \mathcal{C}$, $uu^T = L(uu^T) = Quu^TQ^T = (Qu)(Qu)^T$. By Prop. 6, $Q(u) = \pm u$. Thus, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1 := \{u \in \mathcal{C} : Qu = u\}$ and $\mathcal{C}_2 := \{u \in \mathcal{C} : Qu = -u\}$. Since $\mathcal{C}_1 \setminus \{0\}$ and $\mathcal{C}_2 \setminus \{0\}$ are separated, and

$$\mathcal{C} \setminus \{0\} = \mathcal{C}_1 \setminus \{0\} \cup \mathcal{C}_2 \setminus \{0\},$$

by connectedness of $\mathcal{C} \setminus \{0\}$, we get $\mathcal{C} \subseteq \mathcal{C}_1$ or $\mathcal{C} \subseteq \mathcal{C}_2$, i.e., $Q = \pm I$ on $\mathcal{C}$. If $Q = I$ on $\mathcal{C}$, by linearity, $Q = I$ on $\mathcal{L}^n_+$. As $\mathcal{L}^n_+ - \mathcal{L}^n_+ = R^n$, $Q = I$ on R^n. This yields $L = \widehat{Q} = \widehat{I} = Id$ which is a contradiction. Similarly, if $Q = -I$ on $\mathcal{C}$, we get $Q = -I$ on R^n and hence $L = Id$, which, once again, is a contradiction. This shows that the mapping $Q \mapsto \widehat{Q} : Aut(\widetilde{\mathcal{C}}) \to Aut(\mathcal{K})$ is not surjective. Thus, statements (a) in Theorem 1 and (ii) in Theorem 2 fail to hold.

Now, for $t \in (-1, 1)$, consider the differentiable curve $L(t) : (\mathcal{K}-\mathcal{K}) \oplus (\mathcal{K}-\mathcal{K})^{\perp} \to (\mathcal{K}-\mathcal{K}) \oplus (\mathcal{K}-\mathcal{K})^{\perp}$ given by

$$L(t)(x + y) := x + (1 + t)y \quad (x \in \mathcal{K}-\mathcal{K},\, y \in (\mathcal{K}-\mathcal{K})^{\perp}).$$

Then $L(t) \in Aut(\mathcal{K})$ and $L(0) = Id$. By definition, $L_1 := L'(0) \in Lie(Aut(\mathcal{K}))$. Note that

$$L_1(x) = 0 \text{ for all } x \in \mathcal{K}-\mathcal{K} \quad \text{and} \quad L_1(y) = y \text{ for all } y \in (\mathcal{K}-\mathcal{K})^{\perp}.$$

Suppose, if possible, $L_1 = L_A$ for some $A \in R^{n\times n}$. Since $L_1(\mathcal{K}-\mathcal{K}) = \{0\}$, for any $u \in \mathcal{C}$,

$$0 = L_1(uu^T) = Auu^T + uu^TA^T.$$

It follows that for any $x \in int(\mathcal{L}^n_+)$,

$$x^T(Auu^T + uu^TA^T)x = 0.$$

Thus, $u^Tx\,(x^TAu) = 0$. As $u^Tx > 0$ for any $x \in int(\mathcal{L}^n_+)$ and $0 \neq u \in \mathcal{C}$, we must have $x^TAu = 0$ for all such x and u. Since $\mathcal{L}^n_+ - \mathcal{L}^n_+ = R^n$, $x^TAu = 0$ for all $x \in R^n$ and $u \in \mathcal{C}$; thus, for any $u \in \mathcal{C}$, $Au = 0$. Again, by linearity, $Au = 0$ for all $u \in \mathcal{L}^n_+$ and $Au = 0$ for any $u \in R^n$. Thus, $A = 0$. This gives $L_1 = L_A = 0$. This is not possible, as $L_1(y) = y$ for all $y \in (\mathcal{K}-\mathcal{K})^{\perp}$. Hence, statements (b) in Theorem 1 and (i) in Theorem 2 fail to hold. □

Example 4. Let $\mathcal{C}$ be the closed upper half-plane in R^2. Then $\mathcal{C}$ has nonempty interior, $\mathcal{C} \setminus \{0\}$ is connected, but $\mathcal{C}$ is not pointed. We show that the mappings in Items (i) and (ii) of Theorem 2 are not surjective.

It is clear that every (symmetric) 2×2 matrix that is copositive on $\mathcal{C}$ is also positive semidefinite; hence $\mathcal{E} = \mathcal{S}^2_+$ and so $\mathcal{K} = \mathcal{S}^2_+$. By the result mentioned in the Introduction,

$$Aut(\mathcal{K}) = \left\{ \widehat{Q} : Q \text{ invertible in } \ R^{2\times 2} \right\}$$

and

$$Lie(Aut(\mathcal{K})) = \left\{ L_A : A \in R^{2\times 2} \right\}.$$

For the cone $\mathcal{C}$, it is easily verified that

$$Aut(\mathcal{C}) = \left\{ A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : \ c > 0, a \neq 0 \right\}$$

and

$$Lie(Aut(\mathcal{C})) = \left\{ B = \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} : p, q, r \in R \right\}.$$

Now let

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

For this Q, it is easily verified (using Prop. 4) that $\widehat{Q}$, which is in $Aut(\mathcal{K})$, is not of the form $\widehat{A}$ for any $A \in Aut(\mathcal{C})$. Also, L_Q, which belongs to $Lie(Aut(\mathcal{K})$, is not of the form L_B for any $B \in Lie(Aut(\mathcal{C}))$. $\square$

Example 5. Let $\mathcal{C} = R^2_+ \cup \{\lambda f : \lambda \geq 0\}$, where $f = [-1\,1]^T$. Then $\mathcal{C}$ is pointed, has nonempty interior, but $\mathcal{C} \setminus \{0\}$ is not connected.
Now it is easy to see that $Aut(\mathcal{C}) \subseteq Aut(R^2_+)$ and based on the description of elements in $Aut(R^2_+)$ (see Example 2),

$$Aut(\mathcal{C}) = \left\{ A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} : \alpha > 0 \right\}.$$

Let

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $Q(f) = -f$ and $Q \in Aut(R^2_+)$ and so, $\widehat{Q}(\mathcal{K}) \subseteq \mathcal{K}$. As $Q^{-1} = Q$, we must have $\widehat{Q} \in Aut(\mathcal{K})$. By Prop. 4, $\widehat{Q}$ is not of the form $\widehat{A}$ for any $A \in Aut(\mathcal{C})$. Thus, the mapping in Item (ii) of Theorem 2 is not surjective. $\square$

4. The copositive cone

Recall that $\mathcal{E}$ denotes the copositive cone of $\mathcal{C}$. In this section we describe the elements of $Aut(\mathcal{E})$ and its Lie algebra.

It is easily seen that for any closed cone K in (the real Hilbert space) H,

$$L \in Aut(K) \Leftrightarrow L^* \in Aut(K^*),$$

where L^* denotes the adjoint/transpose of L. This equivalence, along with the equality $(e^{tL})^* = e^{tL^*}$ for any $t \in R$ shows that

$$L \in Lie(Aut(K)) \Leftrightarrow L^* \in Lie(Aut(K^*)).$$

When specialized to a completely positive cone, we get the following.

PROPOSITION 11. *Let $\mathcal{C}$ be any closed cone in R^n. Then*

(i) $L \in Aut(\mathcal{K}) \Leftrightarrow L^* \in Aut(\mathcal{E})$.

(ii) $L \in Lie(Aut(\mathcal{K})) \Leftrightarrow L^* \in Lie(Aut(\mathcal{E}))$.

This proposition, coupled with Theorems 1 and 2, will allow us to describe the automorphisms of $\mathcal{E}$ and the corresponding Lie algebra. Here is a sample result.

COROLLARY 12. *Suppose $\mathcal{C}$ is a closed pointed cone with nonempty interior and $\mathcal{C} \setminus \{0\}$ is connected. Then every $L \in Aut(\mathcal{E})$ is given by*

$$L(X) = Q^T X Q \quad (X \in \mathcal{S}^n)$$

for some $Q \in Aut(\mathcal{C})$ and every $L \in Lie(Aut(\mathcal{E}))$ is of the form $L = L_{A^T}$ for some $A \in Lie(Aut(\mathcal{C}))$.

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