Complementarity properties of Peirce-diagonalizable linear transformations on Euclidean Jordan algebras

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Peirce-diagonalizable linear transformations on a Euclidean Jordan algebra are of the form $L(x) = A \bullet x := \sum_{i\leq j} a_{ij} x_{ij}$, where $A = [a_{ij}]$ is a real symmetric matrix and $\sum x_{ij}$ is the Peirce decomposition of an element $x$ in the algebra with respect to a Jordan frame. Examples of such transformations include Lyapunov transformations and quadratic representations on Euclidean Jordan algebras. Schur (or Hadamard) product of symmetric matrices provides another example. Motivated by a recent generalization of the Schur product theorem, in this article, we study general and complementarity properties of such transformations.

Keywords: Peirce-diagonalizable transformation, linear complementarity problem, Euclidean Jordan algebra, symmetric cone, Schur/Hadamard product, Lyapunov transformation, quadratic representation.

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1. Introduction

Let $L$ be a linear transformation on a Euclidean Jordan algebra $(V, \circ, \langle \cdot, \cdot \rangle)$ of rank $r$ and let $S^r_r$ denote the set of all real $r \times r$ symmetric matrices. We say that $L$ is Peirce-diagonalizable if there exist a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ (with a specified ordering of its elements) in $V$ and a matrix $A = [a_{ij}] \in S^r_r$ such that for any $x \in V$ with its Peirce decomposition $x = \sum_{i \leq j} x_{ij}$ with respect to $\{e_1, e_2, \ldots, e_r\}$, we have

$$L(x) = A \bullet x := \sum_{1 \leq i \leq j \leq r} a_{ij} x_{ij}. \quad (1)$$

The above expression defines a Peirce-diagonal transformation and a Peirce diagonal representation of $L$.

Our first example is obtained by taking $V = S^r_r$ with the canonical Jordan frame (see Section 2 for various notations and definitions). In this case, $A \bullet x$ reduces to the well-known Schur (also known as Hadamard) product of two symmetric matrices and $L$ is the corresponding induced transformation.

On a general Euclidean Jordan algebra, two basic examples are: The Lyapunov transformation $L_a$ and the quadratic representation $P_a$ corresponding to any ele-

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ment $a \in V$. These are, respectively, defined on $V$ by

$$L_a(x) = a \circ x \quad \text{and} \quad P_a(x) = 2a \circ (a \circ x) - a^2 \circ x.$$  

If the spectral decomposition of $a$ is given by

$$a = a_1e_1 + a_2e_2 + \cdots + a_re_r,$$

where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame and $a_1, a_2, \ldots, a_r$ are the eigenvalues of $a$, then with respect to this Jordan frame and the induced Peirce decomposition $x = \sum_{i \leq j} x_{ij}$ of any element, we have [20]

$$L_a(x) = \sum_{i \leq j} (\frac{a_i + a_j}{2})x_{ij} \quad \text{and} \quad P_a(x) = \sum_{i \leq j} (a_i a_j)x_{ij}. \quad (2)$$

Thus, both $L_a$ and $P_a$ have the form (1) and the corresponding matrices are given, respectively, by $[\frac{a_i + a_j}{2}]$ and $[a_ia_j]$. We shall see (in Section 3) that every transformation $L$ given by (1) is a linear combination of quadratic representations.

Our fourth example deals with Löwner functions. Given a differentiable function $\phi : R \rightarrow R$, consider the corresponding Löwner function $\Phi : V \rightarrow V$ defined by (the spectral decompositions)

$$a = \lambda_1e_1 + \lambda_2e_2 + \cdots + \lambda_re_r \quad \text{and} \quad \Phi(a) = \phi(\lambda_1)e_1 + \phi(\lambda_2)e_2 + \cdots + \phi(\lambda_re_r).$$

Then the directional derivative of $\Phi$ at $a = \sum \lambda_ie_i$ in the direction of $x = \sum x_{ij}$ (Peirce decomposition written with respect to $\{e_1, e_2, \ldots, e_r\}$) is given by

$$\Phi'(a; x) := \sum_{i \leq j} a_{ij}x_{ij},$$

where $a_{ij} := \frac{\phi(\lambda_i) - \phi(\lambda_j)}{\lambda_i - \lambda_j}$ (which, by convention, is the derivative of $\phi$ when $\lambda_i = \lambda_j$).

We now note that (for a fixed $a$), $\Phi'(a; x)$ is of the form (1). In [16], Korányi studies the operator monotonicity of $\Phi$ based on such expressions.

Expressions like (1) also appear in connection with the so-called uniform nonsingularity property, see the recent paper [2] for further details.

Our objective in this paper is to study Peirce-diagonalizable transformations, and, in particular, describe their complementarity properties. Consider a Euclidean Jordan algebra $V$ with the corresponding symmetric cone $K$. Given a linear transformation $L$ on $V$ and an element $q \in V$, the (symmetric cone) linear complementarity problem, denoted by $\text{LCP}(L, K, q)$, is to find an $x \in V$ such that

$$x \in K, \quad L(x) + q \in K, \quad \text{and} \quad \langle L(x) + q, x \rangle = 0. \quad (3)$$

This problem is a generalization of the standard linear complementarity problem [3] which corresponds to a square matrix $M \in \mathcal{R}^{n \times n}$, the cone $\mathcal{R}_+^n$, and a vector $q \in \mathcal{R}^n$; it is a special case of a variational inequality problem [4].

In the theory of complementarity problems, global uniqueness and solvability issues are of fundamental importance. A linear transformation $L$ is said to have the $\mathbf{Q}$-property ($\mathbf{GUS}$-property) if for every $q \in V$, $\text{LCP}(L, K, q)$ has a solution (respectively, unique solution); it is said to have the $\mathbf{R}_0$-property if $\text{LCP}(L, K, 0)$ has a unique solution (namely zero). In the context of $L_a$, $\mathbf{GUS}$, $\mathbf{Q}$, and $\mathbf{R}_0$ properties are all equivalent; moreover, they hold if and only if $a > 0$, that is, when $a$ belongs
to the interior of symmetric cone of \( V \). Similarly, for \( P_a \), the properties \( \text{GUS} \), \( Q \), and \( R_0 \) properties are equivalent, see [8]. In this paper, we extend these results to certain Peirce-diagonalizable transformations. Our motivation for this study also comes from a recent result which generalizes the well-known Schur product theorem to Euclidean Jordan algebras [19]: If \( A \) in (1) is positive semidefinite and \( x \in K \), then \( A \cdot x \in K \).

The organization of the paper is as follows: Section 2 covers some basic material. In Section 3, we cover general properties of Peirce-diagonalizable transformations. Section 4 deals with the copositivity property. In this section, we introduce certain cones between the cones of completely positive matrices and doubly nonnegative matrices. Section 5 deals with the \( Z \)-property, and finally in Section 6, we cover the complementarity properties.

2. Some preliminaries


Throughout this paper, we fix a Euclidean Jordan algebra \((V, \circ, \langle \cdot, \cdot \rangle)\) of rank \( r \) with the corresponding symmetric cone (of squares) \( K \) [5], [9]. It is well-known that \( V \) is a product of simple algebras, each isomorphic to one of the matrix algebras \( S^n \), \( H^n \), \( Q^n \) (which are the spaces of \( n \times n \) Hermitian matrices over real numbers/complex numbers/quaternions), \( O^3 \) (the space of \( 3 \times 3 \) Hermitian matrices over octonions), or the Jordan spin algebra \( L^n \) \((n \geq 3)\). In the matrix algebras, the (canonical) inner product is given by

\[
\langle X, Y \rangle = \text{Re} \text{trace}(XY)
\]

and in \( L^n \), it is the usual inner product on \( \mathbb{R}^n \). The symmetric cone of \( S^n \) will be denoted by \( S^n_+ \) (with a similar notation in other spaces). In any matrix algebra, let \( E_{ij} \) be the matrix with ones in the \((i, j)\) and \((j, i)\) slots and zeros elsewhere; we write, \( E_i := E_{ii} \). In a matrix algebra of rank \( r \), \( \{E_1, E_2, \ldots, E_r\} \) is the canonical Jordan frame. We use the notation \( A = [a_{ij}] \) to say that \( A \) is a matrix with components \( a_{ij} \); we also write

\[
A \succeq 0 \iff A \in S^n_+ \quad \text{and} \quad x \geq 0 \ (> 0) \iff x \in K \ (\text{int}(K)).
\]

For a given Jordan frame \( \{e_1, e_2, \ldots, e_r\} \) in \( V \), we consider the corresponding Peirce decomposition ([5], Theorem IV.2.1)

\[
V = \sum_{1 \leq i \leq j \leq r} V_{ij},
\]

where \( V_{ii} = R e_i \) and \( V_{ij} = \{x \in V : x \circ e_i = \frac{1}{2} x = x \circ e_j\} \), when \( i \neq j \). For any \( x \in V \), we get the corresponding Peirce decomposition \( x = \sum_{i \leq j} x_{ij} \). We note that this is an orthogonal direct sum and \( x_{ii} = x_i e_i \) for all \( i \), where \( x_i \in R \).

We recall a few new concepts and notations. Given a matrix \( A \in S^r \), a Euclidean Jordan algebra \( V \) and a Jordan frame \( \{e_1, e_2, \ldots, e_r\} \) with a specific ordering, we define the linear transformation \( D_A : V \to V \) and Schur (or Hadamard) products in the following way: For any two elements \( x, y \in V \) with Peirce decompositions \( x = \sum_{i \leq j} x_{ij} \) and \( y = \sum_{i \leq j} y_{ij} \),

\[
D_A(x) = A \cdot x := \sum_{1 \leq i \leq j \leq r} a_{ij} x_{ij} \in V
\]  (4)
and
\[ x \triangle y := \sum_{i=1}^{r} \langle x_{ii}, y_{ii} \rangle E_i + \frac{1}{2} \sum_{1 \leq i < j \leq r} \langle x_{ij}, y_{ij} \rangle E_{ij} \in S^r. \] (5)

We immediately note that
\[ \langle D_A(x), x \rangle = \langle A \bullet x, x \rangle = \sum_{i \leq j} a_{ij} \| x_{ij} \|^2 = \langle A, x \triangle x \rangle, \] (6)

where the inner product on the far right is taken in \( S^r \).

When \( V \) is \( S^r \) with the canonical Jordan frame, the above products \( A \bullet x \) and \( x \triangle y \) coincide with the well-known Schur (or Hadamard) product of two symmetric matrices. This can be seen as follows. Consider matrices \( x = [\alpha_{ij}] \) and \( y = [\beta_{ij}] \) in \( S^r \) so that \( x = \sum_{i \leq j} \alpha_{ij} E_{ij} \) and \( y = \sum_{i \leq j} \beta_{ij} E_{ij} \) are their corresponding Peirce decompositions with respect to the canonical Jordan frame. Then simple calculations show that
\[ A \bullet x = \sum_{i \leq j} a_{ij} \alpha_{ij} E_{ij} = [a_{ij} \alpha_{ij}] \quad \text{and} \quad x \triangle y = [\alpha_{ij} \beta_{ij}], \]

where the latter expression is obtained by noting that in \( S^r \), \( \langle E_{ij}, E_{ij} \rangle = \text{trace}(E_{ij}^2) = 2 \).

It is interesting to note that
\[ (A \bullet x) \triangle x = A \bullet (x \triangle x), \] (7)

where the ‘bullet’ product on the right is taken in \( S^r \) with respect to the canonical Jordan frame.

**Definition 2.1** A linear transformation \( L \) on \( V \) is said to be **Peirce-diagonalizable** if there exist a Jordan frame and a symmetric matrix \( A \) such that \( L = D_A \).

**Remarks**

1. Note that in the above definition, both \( L \) and \( D_A \) use a specific ordering of the Jordan frame. Any permutation of the elements of the Jordan frame results in a row and column permutation of the matrix \( A \).
2. In the definition of \( D_A \) above, we say that an entry \( a_{ij} \) in \( A \) is **relevant** if \( V_{ij} \neq \{0\} \). (When \( a_{ij} \) is non-relevant, it plays no role in \( D_A \).) By replacing all non-relevant entries of \( A \) by zero, we may rewrite \( D_A = D_B \), where every non-relevant entry of \( B \) is zero. We also note that when \( V \) is simple, every \( V_{ij} \neq \{0\} \) (see [5], Corollary IV.2.4) and thus every entry in \( A \) is relevant.
3. Suppose \( V \) is a product of simple algebras. Then with appropriate grouping of the given Jordan frame, any element \( x \) in \( V \) can be decomposed into components so that the Peirce decomposition of \( x \) splits into Peirce decompositions of components in each of the simple algebras. Because of the uniqueness of the matrix \( B \) (of the previous remark), the transformation \( D_B \) splits into component transformations of the same form, each of which is defined on a simple algebra. Any such component transformation corresponds to a block submatrix of a matrix obtained by permuting certain rows and columns of \( A \). This type of decomposition will allow us, in certain instances, to **restrict our analysis to transformations on simple algebras**.
4. Let \( V \) be a simple algebra and \( \Gamma : V \rightarrow V \) be an algebra automorphism, that is, \( \Gamma \) is an invertible linear transformation on \( V \) with \( \Gamma(x \circ y) = \Gamma(x) \circ \Gamma(y) \) for all
\(x, y \in V\). Since such a \(\Gamma\) preserves the inner product in \(V\) ([5], p. 57), it is easily verified that

\[
\Gamma(x) \triangle' \Gamma(y) = x \triangle y, \tag{8}
\]

where \(\Gamma(x) \triangle' \Gamma(y)\) is computed via the Peirce decompositions with respect to the Jordan frame \(\{\Gamma(e_1), \Gamma(e_2), \ldots, \Gamma(e_r)\}\).

Properties of the products \(A \cdot x\) and \(x \triangle y\) are described in [19] and [11], respectively. In particular, we have the following generalization of the classical Schur’s theorem ([14], Theorem 5.2.1).

**Proposition 2.2** Suppose \(A \succeq 0\) and \(x, y \geq 0\). Then

\[
A \cdot x \geq 0 \quad \text{and} \quad x \triangle y \succeq 0.
\]

It follows that

\[
x = \sum x_{ij} \geq 0 \Rightarrow \|[x_{ij}]\|^2 = x \triangle x \geq 0.
\]

The following proposition is key to many of our results.

**Proposition 2.3** Suppose that \(V\) is simple. Then for any vector \(u \in \mathbb{R}^n_+\), there exists an \(x \in K\) such that

\[
x \triangle x = uu^T. \tag{9}
\]

**Proof.** We assume without loss of generality that \(r \geq 2\) and that the inner product is given by \(\langle x, y \rangle = tr(x \circ y)\) so that the norm of any primitive element is one. (This is because, in a simple algebra, the given inner product is a multiple of the trace inner product, see [5], Prop. III.4.1.) We produce an element \(y \in K\) whose Peirce decomposition with respect to \(\{e_1, e_2, \ldots, e_r\}\) is given by

\[
y = e_1 + e_2 + \cdots + e_r + \sum_{i<j} y_{ij} \quad \text{with} \quad ||y_{ij}|| = \sqrt{2} \quad \text{for all } i < j. \tag{10}
\]

Then, for any \(u = (u_1, u_2, \ldots, u_r)^T \in \mathbb{R}^r_+\), we let

\[
x = \sum_1^r u_i e_i + \sum_{i<j} \sqrt{u_i u_j} y_{ij} = P_a(y),
\]

where \(a = \sum_1^r \sqrt{u_i e_i}\). Since \(P_a(K) \subseteq K\), we see that \(x \in K\). A direct verification proves (9). Now to construct \(y\), we proceed as follows. When \(V\) (which is simple) has rank 2, we take any \(y_{12} \in V_{12}\) with norm \(\sqrt{2}\). Then the statement \(y = e_1 + e_2 + y_{12} \in K\) follows from Lemma 1 in [12] or by the equality \(y^2 = 2y\) (the proof of which requires the use of the identity \(y_{12}^2 = \|[y_{12}]\|^2 (e_1 + e_2)\), see Prop. IV.1.4 in [5]).

When the rank of \(V\) is more than 2, the (simple) algebra \(V\) is isomorphic to a matrix algebra. In this case, we may take the canonical Jordan frame as the given frame (see Remark 4 above) and let \(y_{ij} = E_{ij}\) for all \(i < j\). Then \(y\), which is the matrix of ones, is in \(K\). This completes the proof of the proposition. □
Remark (5) In the construction of \( y \) above (when the rank is more than two), the use of the classification theorem of simple algebras can be avoided by the following argument (which is a slight modification of one) suggested by a referee. 

For \( 1 < i \leq r \), choose \( y_{ij} \in V_{ij} \) with \( \| y_{ij} \| = \sqrt{2} \) and for \( 1 < i < j \leq r \), define \( y_{ij} = 2y_{ij} + y_{ij} \). By Lemma IV.2.2 in [5], \( \| y_{ij} \| = \sqrt{2} \). Let \( y \) as in (10). We claim that \( y^2 = ry \), proving the statement \( y \in K \). To prove \( y^2 = ry \), we induct on \( r \). Write \( y = e_1 + v + w \), where \( v = y_{12} + \cdots + y_{1r}, w = e_2 + e_3 + \cdots + e_r + \sum_{1 \leq i < j \leq r-1} y_{ij} \). Then \( w \) belongs to an appropriate simple algebra of rank \( r-1 \) and so \( w^2 = (r-1)w \). Now using the standard properties of Peirce spaces (specifically, Prop. IV.1.4(i), Theorem IV.2.1(iii), Lemma IV.2.2 in [5]), we get \( e_2^2 = e_1, v^2 = (r-1)e_1 + w, 2e_1 \circ v = v, 2v \circ w = (r-1)v \). This shows that \( y^2 = ry \).

By taking \( u \) to be the vector of ones in the above theorem, and using (7), we get the following.

Corollary 2.4 When \( V \) is simple, for any \( A \in \mathcal{S}^r \), there exists an \( x \in K \) such that

\[
A = (A \bullet x) \triangle x.
\]

3. Some general properties

From now on, unless otherwise specified, Peirce decompositions and \( D_A \) are considered with respect to the Jordan frame \( \{e_1, e_2, \ldots, e_r\} \) in \( V \). Our first result describes some general properties of \( D_A \).

Theorem 3.1 For any \( A \in \mathcal{S}^r \), the following statements hold:

1. \( D_A \) is self-adjoint.
2. The spectrum of \( D_A \) is \( \sigma(D_A) = \{a_{ij} : 1 \leq i \leq j \leq r, a_{ij} \text{ relevant}\} \).
3. \( D_A \) is diagonalizable.
4. If \( \Gamma \) is an algebra automorphism of \( V \), then \( \Gamma^{-1}D_A \Gamma \) is Peirce-diagonalizable with respect to the Jordan frame \( \{\Gamma^{-1}(e_1), \Gamma^{-1}(e_2), \ldots, \Gamma^{-1}(e_r)\} \) and with the same matrix \( A \).
5. \( D_A \) is a finite linear combination of quadratic representations corresponding to mutually orthogonal elements which have their spectral decompositions with respect to \( \{e_1, e_2, \ldots, e_r\} \).
6. If \( A \succeq 0 \), then \( D_A = \sum_1^k P_a \), with \( 1 \leq k \leq r \), where \( a_i \)'s are mutually orthogonal and have their spectral decompositions with respect to \( \{e_1, e_2, \ldots, e_r\} \). In particular, if \( A \) is completely positive (that is, \( A \) is a finite sum of matrices of the form \( uu^T \), where \( u \) is a nonnegative vector), we may assume that \( a_i \in K \) for all \( i \).
7. If \( A \succeq 0 \), then \( D_A(K) \subseteq K \); converse holds if \( V \) is simple.

Proof. (1) This follows from the equality \( \langle D_A(x), y \rangle = \sum_{i \leq j} a_{ij} \langle x_{ij}, y_{ij} \rangle = \langle x, D_A(y) \rangle \).

(2) When \( V_{ij} \neq \{0\} \), any nonzero \( x_{ij} \in V_{ij} \) acts as an eigenvector of \( D_A \) with the corresponding eigenvalue \( a_{ij} \). Hence, all relevant \( a_{ij} \)'s form a subset of \( \sigma(D_A) \). Also, if \( x \) is an eigenvector of \( D_A \) corresponding to an eigenvalue \( \lambda \), by writing the Peirce decomposition of \( x \) with respect to \( \{e_1, e_2, \ldots, e_r\} \) and using the orthogonality of the Peirce spaces \( V_{ij} \), we deduce that \( \lambda \) must be equal to some relevant \( a_{ij} \). This proves that the spectrum of \( D_A \) consists precisely of all relevant \( a_{ij} \).

(3) \( D_A \) is a multiple of the Identity on each \( V_{ij} \). By choosing a basis in each \( V_{ij} \), we get a basis of \( V \) consisting of eigenvectors of \( D_A \). This proves that \( D_A \) is diagonalizable.
(4) Let \( \Gamma \) be an algebra automorphism of \( V \); let \( \Gamma^{-1}(e_i) = f_i \) for all \( i \), so that \( \{f_1, f_2, \ldots, f_r\} \) is a Jordan frame in \( V \). Consider the Peirce decomposition \( y = \sum_{i \leq j} y_{ij} \) of any element \( y \) with respect to \( \{f_1, f_2, \ldots, f_r\} \). Then \( \Gamma(y) = \sum_{i \leq j} \Gamma(y_{ij}) \) is the Peirce decomposition of \( \Gamma(y) \) with respect to \( \{e_1, e_2, \ldots, e_r\} \). Hence, \( D_A(\Gamma(y)) = \sum_{i \leq j} a_{ij} \Gamma(y_{ij}) \). This implies that

\[
\Gamma^{-1} D_A \Gamma(y) = \sum_{i \leq j} a_{ij} y_{ij}.
\]

Thus, \( \Gamma^{-1} D_A \Gamma \) is Peirce-diagonalizable with respect to the Jordan frame \( \{f_1, f_2, \ldots, f_r\} \) and with the matrix \( A \).

(5) Since \( A \in S^+ \), we may write \( A \) as a linear combination of matrices of the form \( uu^T \), where \( u \in \mathbb{R}^n \). From (2), \( (uu^T) \cdot x = \sum_{i \leq j} u_i u_j x_{ij} = P_a(x) \), where \( a = \sum_i u_i e_i \). We see that \( D_A \) is a linear combination of quadratic representations of the form \( P_a \), where \( a \) has its spectral decomposition with respect to \( \{e_1, e_2, \ldots, e_r\} \).

(6) When \( A \succeq 0 \), the expression for \( D_A \) in Item (5) becomes a nonnegative linear combination. By absorbing the nonnegative scalars appropriately, we can write \( D_A \) as a sum of quadratic representations. When \( A \) is completely positive, we may assume that \( u \) (which appears in the proof of Item (5)) is a nonnegative vector. In this case, the corresponding \( a \) belongs to \( K \).

(7) If \( A \succeq 0 \), then by Prop. 2.2, \( D_A(K) \subseteq K \). For the converse, suppose \( V \) is simple. By Cor. 2.4, \( A = (A \cdot x) \Delta x \) for some \( x \in K \). When \( D_A(K) \subseteq K \), by Prop. 2.2, \( A \cdot x \succeq 0 \) and so \( (A \cdot x) \Delta x \succeq 0 \). Thus, \( A \succeq 0 \).

**Corollary 3.2** Suppose \( V \) is simple and for some \( a \in V \), \( L_a(K) \subseteq K \). Then \( a \) is a nonnegative multiple of the unit element.

**Proof.** Using the spectral decomposition of \( a = \sum_i a_i e_i \), we may assume that \( L_a = D_A \), where \( A = \left[ \frac{a_i + a_j}{2} \right] \). Since \( L_a(K) \subseteq K \), by the previous theorem, \( A \) is positive semidefinite. By considering the nonnegativity of any \( 2 \times 2 \) principal minor of \( A \), we conclude that all diagonal elements of \( A \) are equal. This gives the stated result.

4. The copositivity property

In this section, we address the question of when \( D_A \) is copositive on \( K \). Recall that an \( n \times n \) real matrix \( A \) (symmetric or not) is a copositive (strictly copositive) matrix if \( x^T A x \geq 0 \) (\( > 0 \)) for all \( 0 \neq x \in \mathbb{R}_+^n \) and \( D_A \) is copositive (strictly copositive) on \( K \) if \( \langle D_A(x), x \rangle \geq 0 \) (\( > 0 \)) for all \( 0 \neq x \in K \). We shall see that copositivity of \( D_A \) is closely related to the cone of completely positive matrices. We first recall some definitions and introduce some notation. For any set \( \Omega \), let \( \text{cone}(\Omega) \) denote the convex cone generated by \( \Omega \) (so that \( \text{cone}(\Omega) \) is the set of all finite nonnegative linear combinations of elements of \( \Omega \)). Let

\[
\text{COP}_n := \text{The set of all copositive matrices in } S^n,
\]

\[
\text{CP}_n := \text{cone}\{uu^T : u \in \mathbb{R}_+^n\},
\]

\[
\mathcal{NN}_n := \text{The set of all nonnegative matrices in } S^n,
\]

\[
\mathcal{D}_n := \text{The set of all totally positive matrices in } S^n,
\]

\[
\mathcal{D}(K) := \text{cone}\{x \Delta x : x \in K\}.
\]
We note that $CP_n$ is the cone of completely positive matrices and $DNN_n$ is the cone of doubly nonnegative matrices; for information on such matrices, see [1].

**Remark** (6) When $V$ is simple, $D(K)$ is independent of the Jordan frame used to define $x \triangle x$. This follows from (8). It is also independent of the underlying inner product in $V$. A similar statement can be made for general algebras (as they are products of simple ones).

It is well-known, see [1], p. 71, that the cones $COP_n, CP_n$, and $DNN_n$ are closed. We prove a similar result for $D(K)$.

**Proposition 4.1** Let $r$ denote the rank of $V$ and $N = \frac{r(r+1)}{2}$. Then

(i) $D(K) = \{ \sum_{i=1}^{N} x_i \triangle x_i : x_i \in K, 1 \leq i \leq N \}$, 
(ii) $D(K)$ is closed in $S^r$, and 
(iii) $D(K) \subseteq DNN_r$. 
(iv) When $V$ is simple, $CP_r \subseteq D(K) \subseteq DNN_r$ with equality for $r \leq 4$.

**Proof.** (i) This follows from a standard application of Carathéodory’s Theorem for cones, see [1], p. 46. 
(ii) To show that $D(K)$ is closed, we take a sequence $\{z_k\} \subseteq D(K) \subseteq S^r$ that converges to $z \in S^n$. Then, with respect to the canonical Jordan frame in $S^r$, we have $(z_k)_{ij} \to z_{ij}$ for all $i \leq j$. Now, $z_k = \sum_{m=1}^{N} x_m \triangle x_m$ with $x_m \in K$ for all $m$ and $k$. Then

$$(z_k)_{ij} = \sum_{m=1}^{N} (x_m^{(k)} \triangle x_m^{(k)})_{ij} = \sum_{m=1}^{N} \varepsilon_{ij} ||(x_m^{(k)})_{ij}||^2 E_{ij},$$

where $\sum_{i \leq j} (x_m^{(k)})_{ij}$ is the Peirce decomposition of $x_m^{(k)}$ and $\varepsilon_{ij}$ is a positive number that depends only on the pair $(i, j)$. As $(z_k)_{ij}$ converges for each $m$ and $(i, j)$, $(x_m^{(k)})_{ij}$ is bounded as $k \to \infty$. We may assume, as $V_{ij}$s are closed, that $(x_m^{(k)})_{ij}$ converges to, say, $(x_m)_{ij}$ in $V_{ij}$. Defining $x_m := \sum_{i \leq j} (x_m)_{ij}$, we see that $x_m^{(k)} \to x_m$. Then $x_m \in K$ and $x_m^{(k)} \triangle x_m^{(k)} \to x_m \triangle x_m$ for each $m$. We conclude that $z = \sum_{m=1}^{N} x_m \triangle x_m \in D(K)$. Thus $D(K)$ is closed. 
(iii) Every matrix in $D(K)$ is positive semidefinite (from Prop. 2.2) and has nonnegative entries (from the definition of $x \triangle x$). Thus we have the stated inclusion. 
(iv) When $V$ is simple, the inclusion $CP_r \subseteq D(K)$ follows from Prop. 2.3. When $r \leq 4$, it is known, see [1], p. 73, that $CP_r = DNN_r$. 

**Remarks** (7) For $r \geq 5$, we have the inclusions

$$CP_r \subseteq D(S^r) \subseteq D(H^r) \subseteq D(Q^r) \subseteq DNN_r.$$ 

While $CP_r \neq DNN_r$ for $r \geq 5$, see the example given below, it is not clear if the other inclusions are indeed proper. In particular, it is not clear if $x \triangle x$ is completely positive for every $x \in S^r$.

(8) Consider the following matrices:
\[
M = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 \\
0 & 0 & 1 & 2 & 1 \\
1 & 0 & 0 & 1 & 6 \\
\end{bmatrix}
\quad \text{and} \quad
N = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 3 & 1 & 0 \\
0 & 0 & 1 & 4 & 1 \\
1 & 0 & 0 & 1 & 5 \\
\end{bmatrix}.
\]

It is known that \(M\) and \(N\) are doubly nonnegative; while \(N\) completely positive, \(M\) not completely positive, see [1], p. 63 and p. 79. It can be easily shown that \(M\) and \(N\) are not of the form \(x \odot x\) for any \(x \in \mathcal{S}_+^5\). (If \(M = x \odot x\) for some \(x \in \mathcal{S}_+^5\), then the entries of \(x\) are the square roots of entries of \(M\). The leading 3 \times 3 principal minor of any such \(x\) is negative.) It may be interesting to see if \(M\) belongs to \(\mathcal{D}(\mathcal{S}_+^5)\) or not.

**Theorem 4.2** Suppose \(V\) is simple and \(D_A\) is copositive on \(K\). Then \(A\) is a copositive matrix.

**Proof.** Since \(V\) is simple, we can apply Prop. 2.3: For any \(u \in \mathbb{R}^r_+\), there is an \(x \in K\) such that \(x \odot x = uu^T\). Then by (6),

\[
0 \leq \langle D_A(x), x \rangle = \langle A, x \odot x \rangle = \langle A, uu^T \rangle = u^T Au.
\]

Thus, \(A\) is a copositive matrix. \(\square\)

**Remark (9).** It is clear that the converse of the above result depends on whether the equality \(\mathcal{CP}_r = \mathcal{D}(K)\) holds or not.

5. **The \(\mathbf{Z}\)-property**

A linear transformation \(L\) on \(V\) is said to be a \(\mathbf{Z}\)-transformation if

\[
x \geq 0, \quad y \geq 0, \quad \text{and} \quad \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0.
\]

Being a generalization of a \(\mathbf{Z}\)-matrix, such a transformation has numerous eigenvalue and complementarity properties, see [10] and [12]. In this section, we characterize the \(\mathbf{Z}\)-property of a Peirce-diagonalizable transformation.

Corresponding to a Jordan frame \(\{e_1, e_2, \ldots, e_r\}\), we define two sets in \(V\):

\[
\mathcal{C} := \{x \odot y : x \geq 0, y \geq 0\},
\]

and

\[
\hat{\mathcal{C}} := \{x \odot y : 0 \leq x \perp y \geq 0\}.
\]

Now, define the linear transformation \(\Lambda : \mathcal{S}^r \to \mathcal{S}^r\) by \(\Lambda(Z) := EZE = \delta(Z)E\), where \(Z = [z_{ij}], \delta(Z) = \sum_{i,j} z_{ij}\), and \(E\) denotes the matrix of ones. It is clear that \(\Lambda\) is self-adjoint on \(\mathcal{S}^r\).

**Lemma 5.1** When \(V\) is simple with rank \(r\), we have \(\mathcal{C} = \mathcal{S}_+^r\) and \(\hat{\mathcal{C}} = \mathcal{S}_+^r \cap \text{Ker}(\Lambda)\).

**Proof.** From Prop. 2.2, \(\mathcal{C} \subseteq \mathcal{S}_+^r\). To see the reverse inclusion, let \(A \in \mathcal{S}_+^r\). By Cor. 2.4, there is an \(x \in K\) such that \(A = (A \bullet x) \odot x\). By Prop. 2.2, \(y := A \bullet x \geq 0\) and so \(A = y \odot x\), where \(x, y \in K\). Thus, \(\mathcal{S}_+^r \subseteq \mathcal{C}\) and the required equality holds.
Now, let $B = x \triangle y \in \hat{C}$. Then $\delta(B) = \sum_{i,j} b_{ij} = \sum_{i<j} \langle x_{ij}, y_{ij} \rangle = \langle x, y \rangle = 0$ and so $B \in \text{Ker}(\Lambda)$. Since $B \in C = S_{+}^r$, we have $B \in S_{+}^r \cap \text{Ker}(\Lambda)$. For the reverse inclusion, let $B \in S_{+}^r \cap \text{Ker}(\Lambda)$. As $C = S_{+}^r$, we can write $B = x \triangle y$, where $x, y \in K$. Then $\langle x, y \rangle = \delta(B) = 0$ and so $B \in \hat{C}$. This completes the proof. \hfill $\square$

**Theorem 5.2** Let $V$ be simple and $A \in S^r$. Then $D_A$ has the $Z$-property if and only if there exist $B_k \in S_{+}^r$ and a sequence $\alpha_k \in \mathbb{R}$ for which $A = \lim_{k \to \infty} (\alpha_k E - B_k)$. In this case, $D_A = \lim_{k \to \infty} (\alpha_k I - D_{B_k})$.

**Proof.** From the previous lemma, $\hat{C} = S_{+}^r \cap \text{Ker}(\Lambda)$. Then the dual of $\hat{C}$ is given by, see e.g. [1], p. 48, 

$$(\hat{C})^* = S_{+}^r + \text{Ran}(\Lambda),$$

where the overline denotes the closure. Now, $D_A$ has the $Z$-property if and only if $0 \leq x \perp y \geq 0 \Rightarrow \langle D_A(x), y \rangle \leq 0$.

Writing $\langle D_A(x), y \rangle = \sum_{i<j} a_{ij} \langle x_{ij}, y_{ij} \rangle = \text{trace}(AZ)$, where $Z = x \triangle y \in \hat{C}$, we see that $D_A$ has the $Z$-property if and only if $Z \in \hat{C} \Rightarrow \langle A, Z \rangle \leq 0$.

This means that $D_A$ has the $Z$-property if and only if $-A \in (\hat{C})^* = S_{+}^r + \text{Ran}(\Lambda)$. The stated conclusions follow. \hfill $\square$

**Remark (10)** It is known, see [18], that any $Z$-transformation is of the form $\lim_{k \to \infty} (\alpha_k I - S_k)$, where the linear transformation $S_k$ keeps the cone $K$ invariant. Our theorem above reinforces this statement with an additional information that $S_k$ is once again Peirce-diagonalizable.

6. Complementarity properties

In this section, we present some complementarity properties of $D_A$.

**Theorem 6.1** For an $A \in S^r$, consider the following statements.

(i) $D_A$ is strictly (=strongly) monotone on $V$: $\langle D_A(x), x \rangle > 0$ for all $x \neq 0$.

(ii) $D_A$ has the GUS-property: For all $q \in V$, $\text{LCP}(D_A, K, q)$ has a unique solution.

(iii) $D_A$ has the P-property: For any $x$, if $x$ and $D_A(x)$ operator commute with $x \circ D_A(x) \leq 0$, then $x = 0$.

(iv) Every relevant entry of $A$ is positive.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). When $V$ is simple, the reverse implications hold.

**Proof.** The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are well-known and true for any linear transformation, see [9], Theorem 11.

Now suppose (iii) holds. Since $D_A$ is self-adjoint and satisfies the P-property, all its eigenvalues are positive in view of Theorem 11 in [9]. Item (iv) now follows from Theorem 3.1, Item (2). Now assume that $V$ is simple. Then every entry of $A$ is
relevant and so when (iv) holds, we have

\[ \langle D_A(x), x \rangle = \sum_{i \leq j} a_{ij} ||x_{ij}||^2 > 0 \quad \forall \ x \neq 0. \]

This proves that (iv) \Rightarrow (i). \hfill \square

**Remark** (11) Suppose that \( V \) is simple and \( a \in V \). By writing the spectral decomposition \( a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r \), we consider \( L_a \) and \( P_a \) with corresponding matrices \( \left[ \frac{a_i + a_j}{a} \right] \) and \( [a_i, a_j] \). Applying the above result, we get two known results: \( L_a \) has the \( \mathcal{P} \)-property if and only if \( a > 0 \) and \( P_a \) has the \( \mathcal{P} \)-property if and only if \( \pm a > 0 \).

In what follows, we use the notation \( A \in R_0 \) to mean that \( A \) has the \( R_0 \)-property, that is, \( \text{LCP}(A, R_+, 0) \) has a unique solution (namely zero). A similar notation is used for \( D_A \) in relation to \( \text{LCP}(D_A, K, 0) \).

**Theorem 6.2** Suppose \( A \geq 0 \). If \( A \in R_0 \), then \( D_A \in R_0 \). Converse holds when \( V \) is simple.

**Proof.** Assume that \( A \in R_0 \), but \( D_A \notin R_0 \). Then there is a nonzero \( x \in V \) such that \( x \geq 0 \), \( D_A(x) \geq 0 \), and \( \langle D_A(x), x \rangle = 0 \). We have from (6),

\[ 0 = \langle D_A(x), x \rangle = \langle A, x \triangle x \rangle. \]

As \( x \geq 0 \) in \( V \), \( X := x \triangle x \succeq 0 \) (by Prop. 2.2). Now in \( S^r \),

\[ X \succeq 0, A \succeq 0, \quad \text{and} \quad \langle A, X \rangle = 0 \Rightarrow AX = 0. \]

Since \( X \), which is nonzero, consists of nonnegative entries, for any nonzero column \( u \) in \( X \), we have \( Au = 0 \). This \( u \) is a nonzero solution of the problem \( \text{LCP}(A, R_+, 0) \), yielding a contradiction. Hence, \( D_A \in R_0 \).

For the converse, suppose that \( V \) is simple and \( D_A \in R_0 \). Assume, if possible, that \( A \notin R_0 \). Then there exists a nonzero \( u \in R^r \) such that

\[ u \geq 0, \quad Au \geq 0, \quad \text{and} \quad \langle Au, u \rangle = 0. \]

As \( V \) is simple, for this \( u \), by Prop. 2.3, there is an \( x \in K \) such that \( x \triangle x = uu^T \). Since \( u \) is nonzero, \( x \) is also nonzero. Now

\[ \langle D_A(x), x \rangle = \langle A, x \triangle x \rangle = \langle A, uu^T \rangle = u^T Au = 0. \]

We also have \( D_A(x) = A \cdot x \geq 0 \). Thus, even in this case, \( D_A \notin R_0 \). Hence, when \( V \) is simple, \( D_A \in R_0 \Rightarrow A \in R_0 \). \hfill \square

**Remark** (12) Theorem 6.2 may not hold if \( A \) is not positive semidefinite. For example, consider the matrices

\[ A = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \]

with the corresponding transformations \( D_A(x) = A \cdot x \) and \( D_C(x) = C \cdot x \) on \( S^2 \). It is easily seen that \( A \) is in \( R_0 \) while \( D_A \notin R_0 \) and \( D_C \) is in \( R_0 \) while \( C \notin R_0 \).
Before formulating our next result, we recall some definitions. Given a linear transformation \( L \) on \( V \), its principal subtransformations are obtained in the following way: Take any nonzero idempotent \( c \) in \( V \). Then the transformation \( T_c := P_c L : V(c, 1) \to V(c, 1) \) is called a principal subtransformation of \( L \), where \( V(c, 1) = \{ x \in V : x \circ c = x \} \). (We note here that \( V(c, 1) \) is a subalgebra of \( V \).)

We say that \( L \) has the completely \( Q \)-property if for every nonzero idempotent \( c \), the subtransformation \( T_c \) has the \( Q \)-property on \( V(c, 1) \). The transformation \( L \) is said to have the \( S \)-property if there exists a \( d > 0 \) in \( V \) such that \( L(d) > 0 \). The completely \( S \)-property is defined by requiring that all principal subtransformations have the \( S \)-property.

It is well known, see e.g., [6], that for a symmetric positive semidefinite matrix \( A \), the \( R_0 \) and \( Q \) properties are equivalent to the strict copositivity property on \( \mathbb{R}^n \). A related result, given by Malik [17] for linear transformations on Euclidean Jordan algebras, says that for a self-adjoint, cone-invariant transformation, \( R_0 \), completely \( Q \), and completely \( S \) properties are equivalent to strict copositivity property. These two results yield the following

**Theorem 6.3** With \( A \succeq 0 \), consider the following statements:

\( (a) \) \( A \) has the \( Q \)-property.
\( (b) \) \( A \) is strictly copositive.
\( (c) \) \( A \) has the \( R_0 \)-property.
\( (d) \) \( D_A \) has the \( R_0 \)-property.
\( (e) \) \( D_A \) has the completely \( S \)-property.
\( (f) \) \( D_A \) has the completely \( Q \)-property.
\( (g) \) \( D_A \) is strictly copositive.
\( (h) \) \( D_A \) has the \( Q \)-property.

Then

\( (a) \Leftrightarrow (b) \Leftrightarrow (c) \Rightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f) \Leftrightarrow (g) \Rightarrow (h). \)

**Proof.** As \( A \) is symmetric and positive semidefinite, the equivalence of \( (a) \), \( (b) \), and \( (c) \) is given in [6]; the implication \( (c) \Rightarrow (d) \) is given in the previous result. As \( A \) is positive semidefinite, \( D_A \) (which is self-adjoint) keeps the cone invariant. The equivalence of \( (d) - (g) \) now follows from Malik [17], Theorem 3.3.2 and Corollary 3.3.2. Finally, the implication \( (g) \Rightarrow (h) \) follows from the well-known theorem of Karamardian, see [15].

**Remark** (13) One may ask if the reverse implications hold in the above theorem. Since \( D_A \) involves only the relevant entries of \( A \), easy examples (e.g., \( V = \mathbb{R}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \)) can be constructed in the non-simple case to show that \( (h) \) need not imply \( (a) \). While we do not have an answer to this question in the simple algebra case, in the result below we state some necessary conditions for \( D_A \) to have the \( Q \)-property.

**Proposition 6.4** Suppose \( D_A \) has the \( Q \)-property. Then the following statements hold:

\( (i) \) The diagonal elements of \( A \) are positive.
\( (ii) \) When \( V \) is simple, no positive linear combination of two rows of \( A \) can be zero.
\( (iii) \) If \( V \) has rank 2 and \( A \succeq 0 \), then \( D_A \) has the \( R_0 \)-property.

**Proof.** \( (i) \) Since \( D_A \) has the \( Q \)-property, it has the \( S \)-property, that is, there is a \( u > 0 \) in \( V \) such that \( D_A(u) > 0 \). Writing the Peirce decompositions \( u = \sum_{i \leq j} u_{ij} \)
and \( D_A(u) = \sum_{i \leq j} a_{ij} u_{ij} \), we see that \( u_i > 0 \) and \( a_{ii} u_i > 0 \) for all \( i \), where \( u_{ii} = u_i e_i \). Thus, the diagonal of \( A \) is positive. Now, let \( b := \sum_i (a_{ii}^{-1})^{\frac{1}{2}} e_i \) and consider \( \overline{D}_A = P_b D_A P_b \). It is easily seen that \( \overline{D}_A \) has the \( Q \)-property and Peirce-diagonalizable with the matrix whose entries are \( \sqrt{a_{ii}a_{jj}} \).

Note that the diagonal elements of this latter matrix are one.

(ii) Suppose \( V \) is simple and let, without loss of generality, \( \lambda A_1 + \mu A_2 = 0 \), where \( \lambda \) and \( \mu \) are positive numbers, and \( A_1, A_2 \), respectively, denote the first and second rows of \( A \). Let \( a = \sqrt{\lambda} e_1 + \sqrt{\mu} e_2 + e_3 + e_4 + \cdots + e_r \), and consider \( \overline{D}_A := P_a D_A P_a \). Then \( \overline{D}_A \) has the \( Q \)-property and is Peirce-diagonalizable with a matrix in which the sum of the first two rows is zero. Thus, we may assume without loss of generality that in the given matrix \( A \), the sum of first two rows is zero. As \( V \) is simple, the space \( V_{12} \) is nonzero. Let \( q_{12} \) be any nonzero element in \( V_{12} \). We claim that LCP\( (D_A, K, q_{12}) \) does not have a solution. Assuming the contrary, let \( x \) be a solution to this problem and let \( y = D_A(x) + q_{12} \). Then \( x \geq 0 \), \( y \geq 0 \), and \( x \circ y = 0 \). Writing the Peirce decompositions of \( x \) and \( y \) and using the properties of the spaces \( V_{ij} \) (see [5], Theorem IV.2.1), we see that

\[
0 = (x \circ y)_{12} = \frac{1}{2} (a_{11} + a_{12}) x_1 x_{12} + \frac{1}{2} (a_{12} + a_{22}) x_2 x_{12} + \sum_{2 < i} (a_{1i} + a_{2i}) x_i + x_{2i} + \frac{1}{2} (x_1 + x_2) q_{12},
\]

where \( x_{11} = x_1 e_1 \) and \( x_{22} = x_2 e_2 \). Since the sum of first two rows of \( A \) is zero, the above expression reduces to \( 0 = \frac{1}{2} (x_1 + x_2) q_{12} \). As \( q_{12} \) is nonzero, we get \( x_1 + x_2 = 0 \). Now, since \( x \geq 0 \), we must have \( x_1 \geq 0 \) and \( x_2 \geq 0 \). Thus, we have \( x_1 = 0 = x_2 \). This leads, from \( x \geq 0 \), to \( x_{1j} = 0 = x_{2j} \) for all \( j \), see [7], Prop. 3.2. But then \( y \geq 0 \) together with \( y_{11} = a_{11} x_{11} = 0 \) leads to \( y_{1j} = 0 \) for all \( j \), and, in particular, to \( y_{12} = 0 \). Now we see that \( 0 = y_{12} = a_{12} x_{12} + q_{12} = q_{12} \), which contradicts our assumption that \( q_{12} \) is nonzero. Hence, \( D_A \) does not have the \( Q \)-property, contrary to our assumption. This proves the stated assertion.

(iii) Now assume that \( V \) has rank 2. In this case, \( A \) is a \( 2 \times 2 \) matrix. Without loss of generality (see the proof of Item (i)), assume that the diagonal of \( A \) consists of ones. Given that \( D_A \) has the \( Q \)-property, we claim that \( A \) has the \( R_0 \)-property and (via the previous theorem) \( D_A \) has the \( R_0 \)-property. Assume, if possible, that there is a nonzero vector \( d \in \mathcal{R}^2 \) such that \( d \geq 0 \), \( Ad \geq 0 \), and \( \langle d, Ad \rangle = 0 \). As \( A \) is symmetric and positive semidefinite, these conditions lead to \( Ad = 0 \). Since each diagonal element of \( A \) is one, \( d \) cannot have just one nonzero component. Thus, both components of \( d \) are nonzero. In this case, a positive linear combination of rows of \( A \) would be zero. This implies, see Item (ii), that \( V \) must be non-simple. In this case, \( V \) is isomorphic to \( \mathcal{R}^2 \) and we may write for any \( x, x = x_1 e_1 + x_2 e_2 \), \( D_A(x) = a_{11} x_1 e_1 + a_{22} x_2 e_2 \). Thus, we may regard \( D_A \) as a diagonal matrix acting on \( \mathcal{R}^2 \). Now, it is easy to show (from the \( Q \)-property) that \( D_A \) has the \( R_0 \)-property. This completes the proof.

In the next two remarks, we show how to recover known results from the above proposition.

Remarks (14) Suppose for some \( a \in V \), \( D_A = L_a \) has the \( Q \)-property. By writing the spectral decomposition \( a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r \), we may let \( A = \frac{a_1 + a_2}{2} \). By Item (i) of the above proposition, \( a_i > 0 \) for all \( i \), that is, \( a > 0 \). Then, \( \langle L_a(x), x \rangle = \langle a, x^2 \rangle > 0 \) for all \( x \neq 0 \). Thus, \( L_a \) is strictly monotone.
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