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INVERSE AND IMPLICIT FUNCTION THEOREMS FOR *H*-DIFFERENTIABLE AND SEMISMOOTH FUNCTIONS

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With utmost pleasure, I dedicate this article to my teacher, mentor, and friend Olvi Mangasarian on the occasion of his 70th birthday

In this article, we prove inverse and implicit function theorems for H-differentiable functions, thereby giving a unified treatment of such theorems for C^1-functions, PC^1-functions, and for locally Lipschitzian functions. We also derive inverse and implicit function theorems for semismooth functions.

Keywords: H-differentiability; Locally Lipschitzian functions; Semismooth; Generalized Jacobian; Degree; Coherent orientation

1 INTRODUCTION

In this article, we present inverse and implicit function theorems for H-differentiable functions and semismooth functions, thereby generalizing the classical inverse and implicit function theorems to certain classes of nonsmooth functions. The classical inverse function theorem [47] asserts that a continuously differentiable function $f\colon R^n \to R^n$ is locally invertible at a point with a continuously differentiable inverse if the (Fréchet) derivative of the function at that point is nonsingular. The corresponding implicit function theorem says that when a continuously differentiable function $f(x, y)$ vanishes at a point (x^*, y^*) with $f_x^{'}(x^*, y^*)$ nonsingular, the equation $f(x, y) = 0$ can be solved for y in terms of x in a neighborhood of (x^*, y^*). There are numerous inverse and implicit function theorems in the literature. The classical inverse and implicit function theorems have been extended in various directions, e.g., to Banach spaces [1], to multivalued mappings [8,38], to nonsmooth functions [4,32,37,55], etc. Since our setting here is finite dimensional and our functions are single valued, we describe only those generalizations of the classical results that are relevant to our discussion. In 1976, Clarke extended the classical inverse and implicit function theorems to locally Lipschitzian functions by considering the so-called generalized Jacobian; see Refs. [37,55] for

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similar results. In 1991, Robinson [44] proved an implicit function theorem for nonsmooth functions based on the concept of strong approximation and on a certain (numerical) measure of Lipschitzian invertibility. In 1991, Kummer [25] gave a characterization of (local) Lipschitzian invertibility of a function. He showed that a locally Lipschitzian function $f\colon R^n \to R^n$ is Lipschitzian invertible at a point x^* if and only if $0 \notin \Delta f(x^*, d)$ for all $0 \neq d \in R^n$, where $\Delta f(x^*, d)$ consists of all limit points of sequences of the form $\lambda_k^{-1}[f(x^k + \lambda_k d) - f(x^k)]$ with $x^k \to x^*$ and $\lambda_k \downarrow 0$. Since the technical assumptions in these works rely on analytical quantities and/or approximations, we shall refer to these results as analytical inverse/implicit function theorems. Furthermore, the proofs in these works are based on fixed point theorems and/or optimality conditions for minimization problems. In contrast to these, our results in this article are algebraic in nature and rely on topological degree theory. In a rough sense, they say the following: If

(a) $f\colon R^n \to R^n$ is continuous in a neighborhood of a point x^*,
(b) a certain set of $n \times n$ matrices associated with f are coherently oriented (i.e., they have the same nonzero determinantal sign), and
(c) the (topological) index of f at x^* is ± 1,

then f is locally invertible at x^* with the inverse enjoying certain properties that are similar to those of f.

To motivate our results, we begin with the linear complementarity problem LCP(M, q) [7] of finding a vector x in R^n such that

$$x \geq 0, \quad y = Mx + q \geq 0, \quad \text{and} \quad \langle x, y \rangle = 0,$$

where $M \in R^{n\times n}$ and $q \in R^n$. In this context, it is well known [24,30,48] that the piecewise affine function $f(x) := Mx^+ - x^-$, where $x^+ = \max\{x, 0\}$ and $x^- := x^+ - x$, is locally invertible at the origin (equivalently, globally invertible on R^n) if and only if the matrices that describe f are coherently oriented. (This translates to the **P**-property of M, namely, all the principal minors of M are positive.) Generalizing this, Robinson [45] has shown that for an affine variational inequality problem AVI(M, K, q), where $M \in R^{n\times n}$, $q \in R^n$, and K is a polyhedral convex set in R^n, the (piecewise affine) normal map

$$f(x) := M(\Pi_K(x)) + x - \Pi_K(x),$$

where $\Pi_K(x)$ denotes the projection of x onto K, is globally invertible on R^n if and only if the matrices describing f are coherently oriented. Motivated by applications to electrical networks, optimization, etc., various researchers have studied the global homeomorphism properties of piecewise affine mappings; see, e.g., Refs. [3,14,23,24,31,49,50], etc. These global results can be localized to yield local inversion theorems for piecewise affine functions. For example, the local version of a result [Theorem 4.5, Ref. 14] says that a piecewise affine function from R^n to itself is locally invertible at a point if and only if the matrices describing the function at that point are coherently oriented and the topological index of the function at that point is ± 1. In 1996, Pang and Ralph [35] described a generalization of this to piecewise smooth (PC^1) functions. In that paper, it was shown that if at the point under consideration, the so-called Bouligand (sub)differential of the function is coherently oriented and if the index of the function is ± 1, then the function is locally invertible with an inverse that is piecewise smooth; see Ref. [27] for a related implicit function theorem, and Ref. [43] for inverse/implicit function theorems for PC^r functions.

In recent times, a class of functions called semismooth functions have attracted a lot of attention in the optimization community; see, for example, Refs. [28,34,39,42] and Section 2.3

for further references. Since piecewise smooth functions are semismooth, it is natural to ask whether the Pang–Ralph result mentioned above is valid for semismooth functions. We show in this article that this is indeed the case. In fact, by utilizing the concepts of H-differentiability and H-differentials, we prove inverse/implicit function theorems which, when specialized, yield classical inverse/implicit function theorems, Clarke inverse/implicit function theorems, Pang–Ralph inverse function theorem, and inverse/implicit function theorems for semismooth functions. It should be noted that our definition of semismoothness deviates from the original definition of Refs. [28,39,42] in that we do not assume directional differentiability on the function. Because of this, our inverse function theorem for semismooth functions (see Corollary 4) makes no mention of the directional differentiability of the inverse function. In a related work, assuming our definition of semismoothness, Sun [52] has shown that the local inverse of a semismooth function (if exists) is semismooth. In a recent paper, Pang *et al.* [36] address the local invertibility of a semismooth function (as given in the original definition). Based on the technical report version [15] of this article and a result of Ref. [49], they prove a complete inverse function theorem for semismooth functions and show, in particular, that if directional differentiability is assumed then the inverse function is also directionally differentiable. As an application, they study strong regularity/stability of isolated solutions of semidefinite cone and Lorentz cone complementarity problems.

The concepts of H-differentiability and H-differential of a function were introduced in Ref. [16] to study univalence properties of nonsmooth functions. It has been observed in Ref. [16] that the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function [5], the Bouligand differential of a semismooth function [39], and the C-differential of a C-differentiable function [40] are examples of H-differentials. H-differentials are related to the approximate Jacobians of Jeyakumar and Luc [19], in that the closure of an H-differential is an approximate Jacobian. Applications of H-differentiability to optimization, complementarity, and variational inequalities are treated in Ref. [54] and characterizations of $\mathbf{P}_0$- and $\mathbf{P}$-properties of a function can be found in Ref. [51].

2 PRELIMINARIES

Throughout this article, $\|x\|$ denotes the (Euclidean) norm of a vector and Ω denotes an open set in an Euclidean space. If a function $f\colon R^n \to R^m$ is Fréchet differentiable at x^*, we denote the Fréchet derivative (Jacobian matrix) by $f'(x^*)$. For a set E in R^n, we denote the interior (closure, boundary, convex hull) of E by int E (respectively, $\overline{E}$, ∂E, co E).

2.1 *H*-differentiability and *H*-differentials

We recall the following from Ref. [16].

DEFINITION 1 *Let $f\colon \Omega \to R^m$ where $\Omega \subseteq R^n$ is an open set and $x^* \in \Omega$. We say that a nonempty subset $T(x^*)$ of $R^{m\times n}$ is an H-differential of f at x^* if for every sequence $\{x^k\}$ converging to x^*, there exists a subsequence $\{x^{k_j}\}$ and a matrix $A \in T(x^*)$ such that*

$$f(x^k j) - f(x^*) - A(x^{k_j} - x^*) = \mathrm{o}(\|x^{k_j} - x^*\|). \tag{1}$$

We say that f is H-differentiable at x^* if f has an H-differential at x^*.
Sometimes, to show the dependence of $T(x^*)$ on f, we write $T_f(x^*)$ instead of $T(x^*)$.

A useful equivalent definition of an H-differential $T(x^*)$ is: for any sequence $x^k := x^* + t_k d^k$ with $t_k \downarrow 0$ and $\|d^k\| = 1$ for all k, there exist convergent subsequences $t_{k_j} \downarrow 0$ and

$d^{k_j} \to d$, and $A \in T(x^*)$ such that

$$\lim_{j\to\infty} \frac{f(x^* + t_{k_j} d^{k_j}) - f(x^*)}{t_{k_j}} = Ad. \tag{2}$$

We note that f is H-differentiable at x^* if and only if there is an open neighborhood U of x^* and a positive constant γ such that

$$\|f(x) - f(x^*)\| \leq \gamma \|x - x^*\| \quad \text{for all } x \in U. \tag{3}$$

(When (3) holds, we may take $R^{m\times n}$ as an H-differential of f at x^*.) It is clear that an H-differential is not unique (since any superset of an H-differential is another H-differential) and that H-differentiability implies continuity.

2.2 Locally Lipschitzian Functions

Let Ω be an open set in R^n. Consider a function $f\colon \Omega \to R^m$ that is locally Lipschitzian on Ω (so that on some neighborhood of each point of Ω, f is Lipschitzian). We recall two well-known properties of such a function; the first one is the famous Rademacher's theorem [see Ref. 6] and the second one follows easily from the definitions [see the proof of Lemma 3.2.2 in Ref. 26].

- f is Fréchet differentiable almost everywhere (in the Lebesgue sense) in Ω.
- If S is a set of (Lebesgue) measure zero in Ω, then $f(S)$ has measure zero.

Let Ω_f denote the set of all points in Ω where f is Fréchet differentiable. Then, at any $x^* \in \Omega$, the (Clarke) generalized Jacobian [5]

$$\partial f(x^*) = \text{co}\{\lim f'(x^k)\colon x^k \longrightarrow x^*, x^k \in \Omega_f\} \tag{4}$$

exists and is nonempty, compact, and convex. The set

$$\partial_{\mathrm{B}} f(x^*) := \{\lim f'(x^k) : x^k \longrightarrow x^*, x^k \in \Omega_f\} \tag{5}$$

is called the Bouligand (sub)differential of f at x^*. We note that the (multivalued) maps $x \mapsto \partial f(x)$ and $x \mapsto \partial_{\mathrm{B}} f(x)$ are both compact valued and upper semicontinuous at every point [5]. The following fact has been shown in Ref. [16] for a locally Lipschitzian function f:

- f is H-differentiable at x^* with $\partial f(x^*)$ as an H-differential.

2.3 Semismooth Functions

Consider a function $f\colon \Omega \subseteq R^n \to R^m$ that is locally Lipschitzian on Ω. We say that f is *semismooth* at $x^* \in \Omega$ if for any $x^k \to x^*$ and $V_k \in \partial f(x^k)$,

$$f(x^k) - f(x^*) - V_k(x^k - x^*) = \text{o}(\|x^k - x^*\|).$$

The notion of semismoothness was introduced in Ref. [28] for functionals and extended to vector functions in Ref. [39]. We note that in these works the directional differentiability of f at x^* is assumed in addition to the above condition. Convex functions and smooth functions are examples of semismooth functions. Scalar products, sums, compositions of semismooth functions are semismooth. Because of their appearance in many equation-based reformulations of nonlinear complementarity and variational inequality problems, semismooth functions have attracted a lot of attention in the optimization community in recent times; see, for example, Refs. [9–13,21,22,34,39,41,42,56], etc.

We make two observations regarding semismooth functions.

- In the above definition of semismoothness, we can replace $\partial f(x^k)$ by the smaller set $\partial_B f(x^k)$.
- If f is semismooth (in particular, piecewise smooth (PC^1), i.e., f is a continuous selection of a finite number of C^1-functions) at x^*, then it is H-differentiable at x^* with $\partial_B f(x^*)$ as an H-differential.

The first observation can be seen by noting (thanks to Caratheodory's theorem [46]) that each matrix in $\partial f(x^k)$ is a convex combination of at most $mn + 1$ matrices in $\partial_B f(x^k)$. The second observation has been proved in Ref. [16]. We note that if f is locally Lipschitz but not semismooth, then $\partial_B f(x^*)$ need not be an H-differential; see, for example, Problem 2.9.16 in [6].

We now formulate the definition of semismoothness of a function with respect to an H-differentiable map.

DEFINITION 2 *Consider a function $f\colon \Omega \subseteq R^n \to R^m$ that is H-differentiable at each point x of Ω with the corresponding H-differential $T(x)$. We say that f is semismooth at $x^* \in \Omega$ with respect to the map $x \mapsto T(x)$ if for any $x^k \to x^*$ and $V_k \in T(x^k)$,*

$$f(x^k) - f(x^*) - V_k(x^k - x^*) = o(\|x^k - x^*\|).$$

We say that f is semismooth on Ω with respect to the map $x \mapsto T(x)$ if f is semismooth at each point of Ω with respect to T.

2.4 Mean Value Theorem for *H*-differentiable Functions

In this subsection, we describe a mean value theorem for H-differentiable functions and an important consequence that is needed in the main result of Section 3.

Following Jeyakumar and Luc [19], we define the concept of an approximate Jacobian: given $f\colon \Omega \subseteq R^n \to R^m$ and $x^* \in \Omega$, we say that a closed set $\partial^* f(x^*) \subseteq R^{m\times n}$ is an approximate Jacobian of f at x^* if for all $v \in R^m$, the following inequality holds:

$$(vf)^-(x^*, u) \le \sup_{M\in\partial^* f(x^*)} \langle Mu, v\rangle \quad \forall u \in R^n,$$

where

$$(vf)^-(x^*, u) := \liminf_{t\downarrow 0} \left\langle v, \frac{f(x^* + tu) - f(x^*)}{t} \right\rangle$$

is the lower Dini derivative of vf at x^* in the direction of u.

We note that

- if a function f is H-differentiable at x^* with $T(x^*)$ as an H-differential, then the closure of $T(x^*)$ is an approximate Jacobian of f at x^*.

To see this, fix any $u \neq 0$. Starting with the sequence $x^k := x^* + (1/k)u$, we produce a subsequence x^{k_j} and a matrix $A \in T(x^*)$ such that

$$Au = \lim_{j\to\infty} k_j \left[f\left(x^* + \frac{1}{k_j} u \right) - f(x^*) \right].$$

It follows that

$$(vf)^-(x^*, u) \leq \langle v, Au \rangle \leq \sup_{M \in \overline{T(x^*)}} \langle v, Mu \rangle$$

proving the claim.

By specializing the results in Ref. [19] we get the following theorem.

THEOREM 1 *Suppose* $f\colon \Omega \subseteq R^n \to R^m$ *is* H*-differentiable with an* H*-differential* $T(c)$ *at each point* c *of the interval* $[a, b] \subseteq \Omega$, *where* Ω *is an open set. Then*

$$f(b) - f(a) \in \overline{\mathrm{co}} \left[\cup_{c \in [a,b]} T(c) \right] (b - a). \tag{6}$$

As a consequence of the above result [see also, Theorem 3.5, Ref. 19], we have the following corollary.

COROLLARY 1 *Suppose* $f\colon \Omega \subseteq R^n \to R^m$ *is* H*-differentiable such that the* H*-differential map* $x \mapsto T(x)$ *is compact valued and upper semicontinuous. Then* f *is locally Lipschitzian on* Ω.

Remark We note that the above corollary fails without the compact, upper semicontinuity property of the multivalued mapping $x \mapsto T(x)$: consider on R,

$$f(x) = x \sin \frac{1}{x} \ \text{ for } x \neq 0 \quad \text{and} \quad f(0) = 0.$$

Then f is H-differentiable on R with

$$T(0) = [-1, 1] \quad \text{and} \quad T(c) = \left\{ \sin \frac{1}{c} - \frac{1}{c} \cos \frac{1}{c} \right\} \ \text{ for } c \neq 0.$$

We see that f is neither locally Lipschitzian nor Fréchet differentiable at zero. In addition, f is not directionally differentiable at zero.

2.5 Some Properties of (Topological) Degree

Given a continuous function $f\colon \overline{\Omega} \subseteq R^n \to R^n$, where Ω is a bounded open set and $p \in R^n \backslash f(\partial\Omega)$, we denote the (topological) degree of f on Ω at p by $\deg(f, \Omega, p)$ [26]. We list below some properties of the degree [17,26] that are relevant to our discussion. In what follows, we denote the distance between p and $f(\partial\Omega)$ by dist $(p, f(\partial\Omega))$.

- If $\deg(f, \Omega, p) \neq 0$, then there is a solution to the equation $f(x) = p$ in Ω.
- (Nearness property) Suppose that $\deg(f, \Omega, p)$ is defined. If $\|q - p\| < \mathrm{dist}(p, f(\partial\Omega))$, then $\deg(f, \Omega, q)$ is defined and is equal to $\deg(f, \Omega, p)$.
- If $\deg(f, \Omega, p)$ is nonzero, then $p \in \mathrm{int} f(\Omega)$.

- Suppose that $\deg(f, \Omega, p)$ is defined. Let u be an isolated solution of the equation $f(x) = p$ in Ω. Let U be any open subset of Ω containing u such that u is the only solution of $f(x) = p$ in $\overline{U}$. Then $\deg(f, U, p)$ is independent of the open set U. We call this common degree the *index of f* at u and denote it by $\text{index}(f, u)$. If f is differentiable at x^* with a nonsingular Jacobian matrix $f'(x^*)$, then $\text{index}(f, x^*) = \text{sgn}\det f'(x^*)$. Moreover, if the equation $f(x) = p$ has a finite number of solutions $u_1, u_2, \ldots, u_l$ in Ω, then

$$\deg(f, \Omega, p) = \sum_1^l \text{index}(f, u_j). \tag{7}$$

- (Homotopy invariance) Suppose that f is homotopic to g on $\overline{\Omega}$, i.e., there exists a continuous function $H_t(x)$: $[0, 1] \times \overline{\Omega} \to R^n$ such that $H_0(x) = f(x)$ and $H_1(x) = g(x)$ for all a $x \in \overline{\Omega}$. If $p \notin H_t(\partial\Omega)$ for all $t \in [0, 1]$, then

$$\deg(f, \Omega, p) = \deg(g, \Omega, p).$$

2.6 A Local Error Bound Result

PROPOSITION 1 *Suppose Ω is an open subset of R^n and $f: \Omega \to R^n$ has an H-differential $T(a)$ at $a \in \Omega$ consisting of nonsingular matrices. Let $\mathcal{S} := \{x \in \Omega: f(x) = f(a)\}$. Then, there exist a positive constant L and a neighborhood U of a such that*

$$\text{dist}(x, \mathcal{S}) \leq \|x - a\| \leq L\|f(x) - f(a)\| \quad (\forall x \in U).$$

In particular, a is an isolated point of the equation $f(x) = f(a)$.

Proof Since the first inequality in the stated conclusion holds trivially, we prove the second inequality. Assuming the contrary, we consider a sequence $x^k \to a$ such that

$$\|x^k - a\| > k\|f(x^k) - f(a)\|$$

for all k. By working with a subsequence if necessary, we let $d := \lim(x^k - a)/(\|x^k - a\|)$ and

$$\lim \frac{f(x^k) - f(a)}{\|x^k - a\|} = Ad$$

for some $A \in T(a)$. We see that $\|d\| = 1$ and $Ad = 0$ contradicting the nonsingularity of matrices in $T(a)$. Hence the conclusion. ■

As an illustration of the above result, we describe a local error bound result for the nonlinear complementarity problem (NCP). Recall that for a given mapping ϕ: $R^n \to R^n$, the nonlinear complementarity problem NCP(ϕ) is to find a vector $x \in R^n$ such that

$$x > 0, \quad \phi(x) \geq 0, \quad \text{and} \quad \langle x, \phi(x)\rangle = 0.$$

It is well known and easy to see that this problem is equivalent to finding a zero of the function

$$f(x) := \min\{x, \phi(x)\}.$$

Suppose

$$f(a) = \min\{a, \phi(a)\} = 0$$

and that ϕ is H-differentiable at a with an H-differential $T_\phi(a)$. It has been shown in Example 8 of Ref. [54] that

$$\{VA + W\colon A \in T_\phi(a),\ V = \mathrm{diag}(v_i),\ W = \mathrm{diag}(w_i),\ V + W = I \text{ and } v_i, w_i \in \{0, 1\}\ \forall i\}$$

is an H-differential of f at a. A modification of the proof of this assertion reveals that the following smaller set is also an H-differential of f at a:

$$T_f(a) := \{VA + W\colon (A, V, W) \in \Gamma\},$$

where Γ consists of all triples (A, V, W) with $A \in T_\phi(a)$, $V = \mathrm{diag}(v_i)$ and $W = \mathrm{diag}(w_i)$, and

$$\begin{aligned} v_i &= 1 \quad \forall i \in \alpha := \{i\colon a_i > \phi_i(a)\}, \\ w_j &= 1 \quad \forall j \in \beta := \{j\colon a_j < \phi_j(a)\}, \\ v_i, w_i &\in \{0, 1\}, \quad v_i + w_i = 1 \qquad \forall i \in \gamma := \{i\colon a_i = \phi_i(a)\}. \end{aligned}$$

When each matrix in $T_f(a)$ is nonsingular, the above proposition gives a local error bound result for $\min\{x, \phi(x)\}$. To further specialize, let ϕ be Fréchet differentiable at a in which case we can let $T_\phi(a) = \{A\}$, where $A = \phi'(a)$. Then each matrix in $T_f(a)$ is nonsingular if and only if each submatrix of A of the form $A_{\delta\delta}$, where $\alpha \subseteq \delta \subseteq \alpha \cup \gamma$, is nonsingular. This condition is precisely the concept of b-regularity (at the point a) introduced in Ref. [33]. Therefore, in the presence of b-regularity at a, we have the inequality

$$\|x - a\| \le \|\min\{x, \phi(x)\}\|,$$

holding for all points x in a suitable neighborhood of a. A statement of this type has been (explicitly) noted in the literature for a continuously differentiable ϕ [18].

Remark In the context of semismooth functions, the nonsingularity of matrices in the H-differential $\partial_B f(a)$ defines the so-called BD-regularity property of f at a [39]. This property has been used in the convergence proofs of many Newton methods for solving semismooth equations.

3 INVERSE FUNCTION THEOREM FOR *H*-DIFFERENTIABLE FUNCTIONS

THEOREM 2 *Let $\Omega \subseteq R^n$ be an open set. Let $f\colon \Omega \to R^n$ be H-differentiable at each point $x \in \Omega$ with an H-differential $T(x)$. Fix a point $x^* \in \Omega$ and suppose that the following conditions hold:*

- (i) *If f is differentiable at an $x \in \Omega$, then $f'(x)T(x)$.*
- (ii) *The multivalued mapping $x \mapsto T(x)$ is compact valued and upper semicontinuous at each point of Ω.*
- (iii) *$T(x^*)$ consists of positively (negatively) oriented matrices.*
- (iv) index$(f, x^*) = 1$ (*respectively*, -1).

Then the following hold:

(a) *f is locally Lipschitzian on Ω.*
(b) *There exist an open neighborhood U of x^* and an open neighborhood V of $f(x^*)$ such that $f: U \to V$ is one-to-one and onto.*
(c) *If $g: V \to U$ is the inverse of f, then g is H-differentiable at each $v \in V$ with an H-differential*

$$R(v) := \{A^{-1}: A \in T(u)\},$$

where $u \in U$ with $f(u) = v$.
(d) *If g is Fréchet differentiable at $v \in V$, then $g'(v) \in R(v)$.*
(e) *The map $v \mapsto R(v)$ is compact valued and upper semicontinuous at each point of V.*
(f) *$g: V \to U$ is locally Lipschitzian.*
(g) *If f is semismooth at $u \in U$ with respect to the map $x \mapsto T(x)$, then g is semismooth at $f(u)$ with respect to the map $y \mapsto R(y)$.*

Remark The condition $f'(x) \in T(x)$ may not automatically hold when f is Fréchet differentiable at x, and $T(x)$ is an arbitrary H-differential of f at x. For, if $f: R^n \to R^n (n \geq 2)$ is Fréchet differentiable at x^*, then $R^{n\times n}\backslash\{f'(x^*)\}$ will also be an H-differential of f at x^*. However, it can be seen from (2) that if f is Fréchet differentiable at x^*, then for any H-differential $T(x^*)$, and any $d \in R^n$, we have

$$F'(x^*)d \in T(x^*)d. \tag{8}$$

Proof Assume that $T(x^*)$ consists of positively oriented matrices and index $(f, x^*) = 1$. Clearly, Item (a) follows from Corollary 1. To see (b), we proceed in several steps.

Step 1 Since $T(x^*)$ consists of positively oriented matrices, we may assume, from Proposition 1, that in an open neighborhood Ω^* of x^*, the equation $f(x) = f(x^*)$ has only one solution, namely, x^*. We may assume, because of condition (ii), that

$$T(x) \text{ consists of positively oriented matrices for all } x \in \Omega^*. \tag{9}$$

Step 2 Fix $\bar{x} \in \Omega^*$. Since $T(\bar{x})$ consists of positively oriented matrices, once again, by Proposition 1, $\bar{x}$ is an isolated solution of $f(x) = f(\bar{x})$ in, say, the closed ball $\overline{B(\bar{x}, \varepsilon)} \subseteq \Omega^*$. We claim that

$$\text{index}(f, \bar{x}) := \deg(f, B(\bar{x}, \varepsilon), f(\bar{x})) > 0. \tag{10}$$

Let S denote the set of all points in $B(\bar{x}, \varepsilon)$, where f is not Fréchet differentiable. Because f is Lipschitzian (item (a)), by Rademacher's theorem, the Lebesgue measure of S is zero. We consider two cases.

Case 1 Suppose $\bar{x} \notin S$. Then f is Fréchet differentiable at $\bar{x}$ and (10) follows from the equality

$$\text{index}(f, \bar{x}) = \text{sgn}\det f'(\bar{x}),$$

condition (i) and (9). Moreover, $f(\bar{x}) \in \text{int}\, f(B(\bar{x}, \varepsilon))$.

Case 2 Suppose that $\bar{x} \in S$. Let $\delta := \mathrm{dist}(f(\bar{x}), f(\partial B(\bar{x}, \in)))$, which we know is positive. It follows from the nearness property that

$$\deg(f, B(\bar{x}, \varepsilon), f(\bar{x})) = \deg(f, B(\bar{x}, \varepsilon), q) \quad \text{for all } q \in B(f(\bar{x}), \delta). \tag{11}$$

Now by the continuity of f, there exists an $\varepsilon_1 \in (0, \varepsilon)$ such that

$$f(B(\bar{x}, \varepsilon_1)) \subseteq B(f(\bar{x}), \delta).$$

By considering a point u in $B(\bar{x}, \varepsilon_1)\backslash S$ (which is nonempty since S has measure zero), we see by Case 1 that, $f(u)$ is an interior point of $f(B(\bar{x}, \varepsilon_1))$; hence $f(B(\bar{x}, \varepsilon_1))$ has a nonempty interior. Now $f(S)$, being the image of a set of measure zero under a locally Lipschitzian function, has measure zero. Thus, we can find a point $p \in [\mathrm{int} f(B(\bar{x}, \varepsilon_1))]\backslash f(S) \subseteq B(f(\bar{x}), \delta)$. Now consider the equation $f(x) = p$. This will not have any solutions on the boundary of $B(\bar{x}, \varepsilon)$ because $||p - f(\bar{x})|| < \delta = \mathrm{dist}(f(\bar{x}), f(\partial B(\bar{x}, \varepsilon)))$. However, it will have a solution in $B(\bar{x}, \varepsilon_1)$ and hence in $B(\bar{x}, \varepsilon)$. Moreover, by (9) and Proposition 1, each solution of $f(x) = p$ in $B(\bar{x}, \varepsilon)$ is isolated. Hence, the equation $f(x) = p$ has a finite number of solutions in $B(\bar{x}, \varepsilon)$. By (7) and (11), we see that $\deg(f, B(\bar{x}, \varepsilon), f(\bar{x})) = \deg(f, B(\bar{x}, \varepsilon), p)$ is the sum of the indexes of f at each of these solution points. Since $p \notin f(S)$, at each of these solution points, f is Fréchet differentiable, and by Case 1, f will have a positive index. We conclude that $\deg(f, B(\bar{x}, \varepsilon), f(\bar{x})) > 0$ proving the claim.

Step 3 Let $r := \mathrm{dist}(f(x^*), f(\partial\Omega^*))$ and define

$$V := B(f(x^*), r) \quad \text{and} \quad U := f^{-1}(V) \cap \Omega^*.$$

We note that U is open and nonempty (since the equation $f(x) = q$, for $q \in V$, has a solution in Ω^* thanks to the nearness property). Since x^* is an isolated solution of $f(x) = f(x^*)$ in Ω^*, we have $\deg(f, U, f(x^*)) = \deg(f, \Omega^*, f(x^*))$. By condition (iv), this degree is 1. We claim that $f\colon U \to V$ is one-to-one. If for a $q \in V$, we have more than one solution in U, then each of these solutions is isolated, and at each such point f will have a positive index by Step 2. By adding these indexes we reach a contradiction to the condition (iv). This proves our claim. Item (b) follows.

(c) Let $g\colon V \to U$ be the inverse of f. Since $f\colon U \to V$ is continuous, one-to-one, and onto, we see (e.g., from the domain of invariance theorem [26]) that f is an open map; the continuity of g follows. Now for the H-differentiability of g, fix a point $v \in V$ and let $R(v) := \{A^{-1}\colon A \in T(u)\}$, where $u = g(v)$. We claim that $R(v)$ is an H-differential of g at v. To see this, let $v^k \to v$ in V. Then, the sequence $u^k := g(v^k)$ converges to u and so by the H-differentiability of f at u, we can find a subsequence u^{k_j} and an $A \in T(u)$ such that

$$f(u^{k_j}) - f(u) - A(u^{k_j} - u) = \mathrm{o}(||u^{k_j} - u||).$$

Since A is invertible, we can easily show (by going through a subsequence of v^{k_j} if necessary) that

$$g(v^{k_j}) - g(v) - A^{-1}(v^{k_j} - v) = -A^{-1}[f(u^{k_j}) - f(u) - A(u^{k_j} - u)] = \mathrm{o}(||v^{k_j} - v||).$$

Since $A^{-1} \in R(v)$, we have our claim. Thus we have (c).

(d) Fix a point v in V and assume that g is Fréchet differentiable at v. By (8) applied to g and R at v, we have $g'(v)d \in R(v)d$ for any vector d. Since each matrix in $R(v)$ is invertible, it

follows that $g'(v)d = 0 \Rightarrow d = 0$ showing that $g'(v)$ is invertible. Writing $v = f(u)$ for some $u \in U$, we see that f is Fréchet differentiable at u and $f'(u) = [g'(v)]^{-1}$. Since $f'(u) \in T(u)$ by condition (i), we see that $g'(v) \in R(v)$, proving (d).
(e) To see this, consider the multivalued map $v \to R(v)$. Since the mapping which takes a nonsingular matrix to its inverse is continuous, we see by condition (ii) that this multivalued map is compact valued and upper semicontinuous.
(f) This follows from (e) and Corollary 1.
(g) Now suppose that f is semismooth at $u \in U$ with respect to T. To see that g is semismooth at $v := f(u)$ with respect to R, consider any sequence $v^k \to v$ and let $v_k \in R(v^k)$ for all k. Without loss of generality, let $v^k \neq v$ for all k so that $u^k \neq u$ for all k. With $V_k = A_k^{-1}$, put

$$\xi^k := f(u^k) - f(u) - A_k(u^k - u)$$

and

$$\eta^k := g(v^k) - g(v) - V_k(v^k - v).$$

Then $\eta^k = -A_k^{-1}\xi^k$. We known that $\xi^k/\|u^k - u\| \to 0$. This yields

$$m := \inf \frac{\|v^k - v\|}{\|u^k - u\|} > 0.$$

(Else, for some subsequence u^{k_j}, $A^{k_j} \to A \in T(u)$ (by condition (ii)) and $d := \lim((u^{k_j} - u)/\|u^k - u\|)$, we have $0 = \lim((f(u^{k_j}) = f(u))/\|u^k - u\|) = Ad$. Since A is invertible and d has norm one, we have a contradiction.) From

$$\frac{\xi^k}{\|v^k - v\|} = \frac{\xi^k}{\|u^k - u\|}\frac{\|u^k - u\|}{\|v^k - v\|} \longrightarrow 0,$$

we get

$$\frac{\|\eta^k\|}{\|v^k - v\|} \le \|A_k^{-1}\| \frac{\|\xi^k\|}{\|u^k - u\|}\frac{1}{m}.$$

Since the sequence A_k^{-1} is bounded above (thanks to (e)), we see that $\eta^k = o(\|v^k - v\|)$ proving (g). ■

Remark When the conditions of the theorem are in place, items (d) and (e) imply that

$$\partial_B g(v) \subseteq R(v) \tag{12}$$

for all $v \in V$.

COROLLARY 2 (The classical inverse function theorem for C^1-functions) *Suppose $f\colon R^n \to R^n$ is C^1 in a neighborhood of $x^* \in R^n$ and that the Fréchet derivative $f'(x^*)$ is nonsingular. Then in a neighborhood of x^*, f admits an inverse g that is C^1 and $g'(f(x^*)) = [f'(x^*)]^{-1}$.*

Proof With $T(x) := \{f'(x)\}$ for all x in a suitable neighborhood of x^*, conditions (i)–(iv) of the above theorem hold. The local inverse g that we get in the above theorem is C^1 in a neighborhood of $f(x^*)$. The equality $g'(f(x^*)) = [f'(x^*)]^{-1}$ comes from items (c) and (d) of the theorem. ■

We recall that a (continuous) function $f\colon \Omega \subseteq R^n \to R^n$ is PC^1 if f is a continuous selection of a finite number of C^1-functions, i.e., there exist $f_1, f_2, \ldots, f_k$, each C^1 on Ω, such that

$$f(x) \in \{f_1(x), f_2(x), \ldots, f_k(x)\} \quad (\forall x \in \Omega).$$

Such a function is known to be semismooth and hence $\partial_B f(x)$ is an H-differential at any point x. In addition, when these functions are essentially active at $x^* \in \Omega$ [35, Lemma 2], we have

$$\partial_B f(x^*) = \{f_j'(x)\colon j = 1, 2, \ldots, k\}.$$

COROLLARY 3 (Inverse function theorem for PC^1-functions [35]) *Let* $f\colon \Omega \subseteq R^n \to R^n$ *be* PC^1 *with selection functions* $f_1, f_2, \ldots, f_k$. *Assume that these functions are essentially active at* $x^* \in \Omega$. *If conditions* (iii) *and* (iv) *of the theorem hold, then* f *has a* PC^1 *inverse in a neighborhood of* x^*.

Proof Since f is semismooth, conditions (i) and (ii) are always true. When conditions (iii) and (iv) of the theorem hold, we have a local inverse g as in the theorem. We see by condition (iii) that each C^1-function f_j is locally invertible at x^* with a C^1 inverse $(f_j)^{-1}$. Clearly, in an appropriate neighborhood of $f(x^*)$, $g(x)$ takes a value in the set $\{(f_j)^{-1}(x)\colon j = 1, 2, \ldots, k\}$. Since g is continuous, we see that g is a PC^1-function. ■

We now specialize the theorem to semismooth functions.

COROLLARY 4 (Inverse function theorem for semismooth functions) *Suppose* $f\colon \Omega \subseteq R^n \to R^n$ *is semismooth. Let* $x^* \in \Omega$ *be such that*

(1) $\partial_B f(x^*)$ *consists of positively* (*negatively*) *oriented matrices*, *and*
(2) index$(f, x^*) = 1$ (*respectively*, -1).

Then f *has a local semismooth inverse at* x^*, i.e., *there exist neighborhoods* U *of* x^* *and* V *of* $y^* := f(x^*)$ *such that* $f\colon U \to V$ *has an inverse* $g\colon V \to U$ *that is semismooth on* V. *Moreover*,

$$\partial_B g(y^*) = \{A^{-1}\colon A \in \partial_B f(x^*)\}.$$

Proof Since f is semismooth, at any point $x \in \Omega$, the set $\partial_B f(x)$ is an H-differential. Moreover, the mapping $x \mapsto \partial_B f(x)$ is compact and upper semicontinuous [5]. Hence, the conditions of the previous theorem are met. Let U, V, and g be as in the theorem. Then g is locally Lipschitzian on V. We claim that g is semismooth at $v \in V$. Since f is semismooth with respect to the map $x \mapsto \partial_B f(x)$, we see that g is semismooth with respect to the map $y \mapsto R(y)$, where $R(y) := \{A^{-1}\colon A \in \partial_B f(x)\}$ and $y = f(x)$. We complete the proof by showing that

$$\partial_B g(v) = \{A^{-1}\colon A \in \partial_B f(u)\}, \tag{13}$$

where $v = f(u)$. The inclusion $\partial_B g(v) \subseteq \{A^{-1}\colon A \in \partial_B f(u)\}$ comes from (12). To see the reverse inclusion, suppose $A \in \partial_B f(u)$. Then $A = \lim f'(u^k)$ where $u^k \to u$. Since A is invertible, $f'(u^k)$ is invertible for large k and $A^{-1} = \lim[f'(u^k)]^{-1}$. We see that g is differentiable at $v^k := f(u^k)$ with $g'(v^k) = [f'(u^k)]^{-1}$, and $A^{-1} = \lim g'(v^k) \in \partial_B g(v)$. We thus have (13). ■

Remark As noted in the introduction, our definition of semismoothness does not require the directional differentiability of the function at the reference point. Suppose in the above Corollary, f is further assumed to be directionally differentiable at x^* so that f is semismooth in the original sense [28,39]. Then, f has a local semismooth inverse (by the above Corollary). In particular, f has a local Lipschitzian inverse. In this setting, Scholtes [49, Theorem 3.2.3] has shown that the inverse of f is directionally differentiable at $f(x^*)$. We conclude that: If f is semismooth in the original sense and satisfies conditions (1) and (2) of the above Corollary, then f has a local inverse that is semismooth in the original sense. This fact and its justification appear in Ref. [36], where the authors, using the (strong) semismoothness properties of the projection maps onto the semidefinite cone and Lorentz cone [2,53] study strong regularity/stability properties of isolated solutions of semidefinite and Lorentz cone complementarity problems.

At this stage, one might ask whether conditions (1) and (2) of the above corollary are necessary for the existence of a local semismooth inverse. The following result answers this question.

THEOREM 3 *Suppose* $f\colon \Omega \subseteq R^n \to R^n$ *is locally Lipschitzian and* $x^* \in \Omega$. *If* f *admits a locally Lipschitzian inverse in a neighborhood of* x^*, *then conditions* (1) *and* (2) *of the previous corollary are satisfied. Moreover, when* f *is semismooth, there is a semismooth inverse of* f *in a neighborhood of* x^* *if and only if conditions* (1) *and* (2) *of the above corollary are satisfied.*

Proof For the first part, suppose there exist neighborhoods U of x^*, V of $y^* = f(x^*)$ such that $f\colon U \to V$ is one-to-one and onto, and the inverse $g\colon V \to U$ is locally Lipschitzian. To see conditions (1) and (2), we proceed in several steps.

First, we show that at any point $u \in U$ where f is Fréchet differentiable, the derivative $f'(u)$ is nonsingular. To see this, fix a point $u \in U$ and assume if possible that $f'(u)d = 0$ for some nonzero vector d. Then

$$\lim \frac{f(u + t_k d) - f(u)}{t_k} = f'(u)d = 0,$$

whenever $t_k \downarrow 0$. This clearly contradicts the following consequence of the Lipschitzian property of g at $f(u)$:

$$\|g(f(u + t_k d)) - g(f(u))\| \leq \alpha \|f(u + t_k d) - f(u)\|$$

(which is valid for some $\alpha > 0$ and for all k large). Hence the nonsingularity of $f'(u)$.

Second, we claim that at any point $u \in U$, $\partial_{\mathrm{B}} f(u)$ consists of invertible matrices. To see this, suppose $A \in \partial_{\mathrm{B}} f(u)$ so that by definition, $A = \lim f'(u^k)$ where $u^k \to u$. We know from the previous argument that $f'(u^k)$ is invertible for all large k. Writing $v^k = f(u^k)$, we see that $[f'(u^k)]^{-1} = g'(v^k) \in \partial_{\mathrm{B}} g(v^k)$. Since the map $x \mapsto \partial_{\mathrm{B}} g(x)$ is compact valued and upper semicontinuous at $v (= \lim v^k)$, we see that a subsequence of $[f'(u^k)]^{-1}$ converges to, say, the matrix B. We conclude that $AB = I$ and that A is invertible.

Third, we establish conditions (1) and (2). Now, since f is one-to-one, $\deg(f, U, f(x^*)) = \pm 1$ [26]. We may assume, without loss of generality, that $\deg(f, U, f(x^*)) = 1$. Since x^* is an isolated solution of $f(x) = f(x^*)$ in U, this degree is index (f, x^*) giving us the condition (2) of the above corollary. In addition, by assuming the connectedness of V (see Step 3 in the proof of Theorem 2) and the homotopy invariance of degree, we may conclude that $\deg(f, U, f(u)) = 1$ for any $u \in U$. As before, this integer is nothing but $\mathrm{index}(f, u)$. Now, if f is Fréchet differentiable at u, then

$$1 = \mathrm{index}(f, u) = \mathrm{sgn} \det f'(u),$$

where we have used the fact that $f'(u)$ is nonsingular. Thus we see that at any differentiable point u, the derivative of f has a positive determinant. Finally any matrix in $\partial_B f(u)$, being invertible and the limit of matrices with positive determinants, has a positive determinant. Thus we have condition (1) of the corollary.

When f is semismooth, the stated conclusion comes from the first part of the theorem and Corollary 4. ■

COROLLARY 5 (Inverse function theorem for locally Lipschitzian functions [4]) *Suppose $f: \Omega \subseteq R^n \to R^n$ is locally Lipschitzian. Let $x^* \in \Omega$ be such that $\partial f(x^*)$ consists of invertible matrices. Then f has a local Lipschitzian inverse g at $y^* = f(x^*)$ with*

(a) $\partial_B g(y^*) \subseteq \{A^{-1}: A \in \partial f(x^*)\}$ *and*
(b) $\partial g(y^*) \subseteq \mathrm{co}\{A^{-1}: A \in \partial f(x^*)\}$.

Proof At each point $x \in \Omega$, we let $T(x) := \partial f(x)$. This is an H-differential of f (see Sec. 2.2); moreover, the map $x \mapsto T(x)$ is compact valued and upper semicontinuous at each point of Ω [5]. The condition (i) of Theorem 2 is clearly satisfied. By convexity of $\partial f(x^*)$, we conclude that all matrices in $\partial f(x^*)$ have the same determinantal sign which is nonzero by the imposed condition. Thus condition (iii) holds. To see condition (iv), fix $A \in \partial f(x^*)$, apply the homotopy invariance of the degree to the homotopy $H_t(x) := t[f(x) - f(x^*)] + (1-t)A(x - x^*)$ to get index $(f, x^*) = \mathrm{sgn}\det A$. The inverse of f obtained in the theorem is locally Lipschitzian. Item (a) follows from (12), and item (b) follows from the definition. ■

Remarks The idea of using topological degree theory to study Lipschitzian invertibility of a Lipschitzian function is not new; see, for example, Ref. [20].

4 IMPLICIT FUNCTION THEOREMS

In this section, based on Theorem 2, we shall derive an implicit function theorem for H-differentiable functions.

Consider a function f of two variables with $x \in R^n$ and $y \in R^m$. For ease of notation, we let

$$z = \begin{pmatrix} x \\ y \end{pmatrix},$$

where x and y are column vectors. (We define z^* and $\bar{z}$ similarly.) If f has an H-differential $T_f(\bar{z})$ at a given point $\bar{z}$, we write any $A \in T_f(\bar{z})$, in the block form as

$$A = \begin{bmatrix} A_x & A_y \end{bmatrix},$$

where A_x and A_y are of sizes $n \times n$ and $n \times m$, respectively. (In the differentiable case, A_x is $f_x(\bar{z})$, the partial derivative of f at $\bar{z}$ with respect to x.)

THEOREM 4 *Consider $f: \mathcal{D} \subseteq R^n \times R^m \to R^n$, where $\mathcal{D}$ is an open set. Suppose f is H-differentiable at each point z of $\mathcal{D}$ with an H-differential given by $T_f(z)$. Let $z^* \in \mathcal{D}$ and assume that*

(a) *$f'(z) \in T_f(z)$ whenever f is Fréchet differentiable at z,*
(b) *the multivalued mapping $z \mapsto T_f(z)$ is compact valued and upper semicontinuous at each point of $\mathcal{D}$,*
(c) *$f(z^*) = 0$,*
(d) *the set $\{A_x: A \in T_f(z^*)\}$ consists of positively (negatively) oriented matrices, and*

(e) index$(h, x^*) = 1$ (*respectively*, -1) *where*

$$h(x) := f\begin{pmatrix} x \\ y^* \end{pmatrix}. \tag{14}$$

Then there exist open sets $U \subseteq R^n \times R^m$ *containing* z^* *and* $W \subseteq R^m$ *containing* y^* *such that to every* $y \in W$, *there is a unique* x *with*

$$\begin{pmatrix} x \\ y \end{pmatrix} = z \in U \quad \text{and} \quad f(z) = 0.$$

If this x *is defined to be* $g(y)$, *then*

(i)

$$f\begin{pmatrix} g(y) \\ y \end{pmatrix} = 0 \quad \forall y \in W;$$

(ii) g *is* H*-differentiable at any* $\bar{y} \in W$ *with an* H*-differential given by*

$$T_g(\bar{y}) = \{-(A_x)^{-1}A_y\colon \begin{bmatrix} A_x & A_y \end{bmatrix} \in T_f(\bar{z})\}.$$

Proof We begin by noting that h, given by (14), is H-differentiable at x^* with

$$T_h(x^*) = \{A_x\colon A = \begin{bmatrix} A_x & A_y \end{bmatrix} \in T_f(z^*)\}.$$

The same conclusion holds at all points near x^*. From condition (d) and Proposition 1, we see that the equation $h(x) = h(x^*)$ has an isolated solution in some neighborhood of x^*. (Because of this, we could define the index of h at x^* in condition (e).)

Now define a function $F\colon \mathcal{D} \to R^n \times R^m$ by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(z) \\ y \end{pmatrix}.$$

It is easily seen that F is H-differentiable at any $\bar{z} \in \mathcal{D}$ with an H-differential given by

$$T_F(\bar{z}) = \left\{ \widehat{A} = \begin{bmatrix} A_x & A_y \\ 0 & I \end{bmatrix} : A = \begin{bmatrix} A_x & A_y \end{bmatrix} \in T_f(\bar{z}) \right\}.$$

We now verify conditions (i)–(iv) of Theorem 2 for F. Clearly, conditions (i) and (ii) of Theorem 2 are satisfied for F and T_F. Since the determinant of any matrix $\widehat{A}$ in T_F is the determinant of the corresponding A_x, by condition (d) above, we see that $T_F(z^*)$ consists of positively (negatively) oriented matrices. We now claim that

$$\text{index}(F, z^*) = \text{index}(h, x^*).$$

We first note that

$$F(z^*) = \begin{pmatrix} 0 \\ y^* \end{pmatrix}.$$

By the nonsingularity of matrices in $T_F(z^*)$, and by Proposition 1, we see that index of F at z^* is well defined. We proceed to justify the above claim.

By our earlier observation, the equation $h(x) = h(x^*) = 0$ has an isolated solution in a neighborhood of x^*. Then, the equation

$$\begin{pmatrix} h(x) \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y^* \end{pmatrix}$$

has a unique solution, namely, z^* in the closure of some open set Ω containing z^*. On $\overline{\Omega}$, we define a homotopy

$$H_t(z) = tF(z) + (1-t)\begin{pmatrix} h(x) \\ y \end{pmatrix} \quad (0 \le t \le 1).$$

On $\overline{\Omega}$, the equation

$$H_t(z) = \begin{pmatrix} 0 \\ y^* \end{pmatrix}$$

leads to $y = y^*$ and $h(x) = 0$, and hence to $z = z^*$. By the homotopy invariance of degree,

$$\deg(H_0, \Omega, F(z^*)) = \deg(H_1, \Omega, F(z^*)).$$

Now

$$H_0(z) = \begin{pmatrix} h(x) \\ y \end{pmatrix} \quad \text{and} \quad H_1(z) = F(z).$$

Since

$$\deg(H_0, \Omega, F(z^*)) = \operatorname{index}(h, x^*) \cdot \operatorname{index}(I, y^*)$$

(where I denotes the identity map) and

$$\deg(H_1, \Omega, F(z^*)) = \operatorname{index}(F, z^*),$$

we have our claim.

At this stage, we have verified the conditions of Theorem 2 for F. We can now find a neighborhood U of z^* and a neighborhood $V \times W$ of $F(z^*)$ such that to each $v \in V$ and $w \in W$, there is a unique $z \in U$ such that

$$F(z) = \begin{pmatrix} v \\ w \end{pmatrix}.$$

In particular, for each $w \in W$, there is a unique $z \in U$ with

$$\begin{pmatrix} f(z) \\ y \end{pmatrix} = F(z) = \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

This implies that $y = w$ and $f(z) = 0$. Thus for each $y \in W$, there is a unique x such that

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \in U$$

and $f(z) = 0$. Letting g be the map that takes y to x, we verify conclusions (i) and (ii) of the theorem. Conclusion (i) is obvious. To get (ii), we proceed as follows. Let G: $V \times W \to U$ denote the inverse of F on U. Then from Theorem 2, we know that G is H-differentiable at any

$$q = \begin{pmatrix} v \\ w \end{pmatrix}$$

with an H-differential given by

$$T_G(q) = \{M^{-1}\colon M \in T_F(z)\},$$

where $F(z) = q$. Now g can be written as

$$g = \Pi_1 \circ G \circ \phi,$$

where ϕ is the inclusion map

$$y \longmapsto \begin{pmatrix} 0 \\ y \end{pmatrix}$$

and Π_1 is the projection map

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x.$$

Thus g, being a composition of H-differentiable maps, is also H-differentiable with an H-differential given by

$$T_g(\bar{y}) = \left\{ \begin{bmatrix} I_x & 0 \end{bmatrix} \begin{bmatrix} A_x & A_y \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_y \end{bmatrix} \colon \begin{bmatrix} A_x & A_y \end{bmatrix} \in T_f(\bar{z}) \right\}.$$

This is seen by observing that both ϕ and Π_1 are Fréchet differentiable, and that straightforward chain rule exists for H-differentiable functions. Now simple algebraic manipulations yield

$$T_g(\bar{y}) = \{-(A_x)^{-1} A_y\colon \begin{bmatrix} A_x & A_y \end{bmatrix} \in T_f(\bar{z})\}.$$

This proves conclusion (ii) and the theorem. ■

Remarks We note that in the above theorem, if f is continuously differentiable (locally Lipschitzian, piecewise smooth, semismooth), then so is g. By specializing the above theorem, we get implicit function theorems for continuously differentiable functions (with $T_f(z) = \{f'(z)\}$), locally Lipschitzian functions (with $T_f(z) = \partial f(z)$), and for piecewise smooth/semismooth functions (with $T_f(z) = \partial_{\mathrm{B}} f(z)$).

5 CONCLUSION

In this article, we have proved inverse and implicit function theorems for H-differentiable functions. Except for some comments on directional differentiability (e.g., Remark following Corollary 4) and some recent references, this article is essentially the same as the technical report [15]. One referee notes that pseudodifferentiable functions, as given in Ref. [29], are H-differentiable and that (a version of) Corollary 1 can be found in Ref. [29].

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