On common linear/quadratic Lyapunov functions for switched linear systems

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- Joint work with Melania Moldovan
- Based on the forthcoming article in

Nonlinear analysis and variational problems,
Eds: P. Pardalos, Th.M. Rassias, and A.A. Khan,
Springer 2009,
Volume dedicated to the memory of George Isac

- Objective is to study switched linear systems via complementarity and duality ideas.

Outline

- Continuous/discrete/positive linear systems
- Complementarity ideas
- **Z**-matrices and transformations
- Switched systems
- Duality ideas
- Positive switched systems
- An extension of Mason-Shorten result
- Miscellaneous

Continuous Linear System

Given $A \in R^{n \times n}$, consider

$$\dot{x} + Ax = 0.$$

Asymptotic stability: $x(t) \to 0$ as $t \to \infty$.

Lyapunov's Theorem (1893):

The following are equivalent:

(1) A is positive stable.

(2) There exists a $P \succ 0$ with $A^T P + PA \succ 0$.

(3) The system $\dot{x} + Ax = 0$ is asymptotically stable.

A is positive stable: Re $\lambda > 0$ for all eigenvalues of A.

$Q(x) := x^T P x$ is a quadratic Lyapunov function.

When A is positive stable, Q decreases along any trajectory of the system.

Discrete Linear System

$$x(k+1) = Ax(k); \;\; k = 1, 2, \ldots .$$

Asymptotic stability: $x(k) \to 0$ as $k \to \infty$.

Stein's Theorem (1952):

The following are equivalent:

(1) A is Schur stable.

(2) There exists a $P \succ 0$ with $P - A^T PA \succ 0$.

(3) The system $x(k+1) = Ax(k)$ is asymptotically stable.

A Schur stable: $|\lambda| < 1$ for all eigenvalues of A.

$Q(x) := x^T P x$ is a quadratic Lyapunov function.

When A is Schur stable, Q decreases along any trajectory of the system.

Positive systems

All trajectories of $\dot{x} + Ax = 0$ are constrained to R_+^n.

This happens if and only if A is a **z**-matrix:

All off-diagonal entries are nonpositive.

If there is $d > 0$ such that $A^T d > 0$, then $Q(x) = x^T d$ is a linear Lyapunov function.

In this case, the system is asymptotically stable.

Linear complementarity problems

$A \in R^{n \times n}$ and $q \in R^n$.

LCP(A, q): Find $x \in R^n$ such that

$$x \geq 0,\ Ax + q \geq 0,\ \text{and } x^T(Ax + q) = 0.$$

A is a **Q**-matrix if LCP(A, q) has a solution for all q.

A is a **P**-matrix if LCP(A, q) has a unique solution for all q or equivalently,

$$x * Ax \leq 0 \Rightarrow x = 0.$$

Semidefinite LCPs

$L : \mathcal{S}^n \to \mathcal{S}^n$ linear, $Q \in \mathcal{S}^n$.

SDLCP(L, Q): Find $X \in \mathcal{S}^n$ such that

$$X \succeq 0,\ L(X) + Q \succeq 0,\ \text{and } \langle X, L(X) + Q \rangle = 0.$$

L has the **Q**-property if SDLCP(L, Q) has a solution for all $Q \in \mathcal{S}^n$.

L has the **P**-property if

X and $L(X)$ commute and $X \circ L(X) \preceq 0$

implies $X = 0$.

LCPs and positive systems

Let A be a **Z**-matrix. Then the following are equivalent:

- There exists $d > 0$ such that $A^T d > 0$.
- A is a **Q**-matrix.
- A is a **P**-matrix.

SDLCPs and linear systems

$A \in R^{n\times n}$, let

$L_A(X) := AX + XA^T$ –the Lyapunov transformation

Gowda-Song (2000): The following are equivalent:

- There exists a $P \succ 0$ such that $L_A(P) \succ 0$.
- A is positive stable.
- L_A has the **Q**-property
- L_A has the $\mathbf{P}$-property.

SDLCPs and discrete linear systems

For $A \in R^{n \times n}$, let

$S_A(X) := X - AXA^T$ –the Stein transformation.

Gowda-Parthasarathy (2000) : The following are equivalent:

- There exists a $P \succ 0$ such that $S_A(P) \succ 0.$
- A is Schur stable (equivalently, S_A is positive stable).
- S_A has the **Q**-property.
- S_A has the **P**-property.

Do L_A and S_A have some sort of **Z**-property?

The z-property

V – Finite dimensional real Hilbert space.

K – proper cone in V: K is a closed convex cone with $K \cap (-K) = \{0\}$.

A linear $L : V \to V$ has the **z**-property if

$$x \in K,\ y \in K^*,\ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0$$

where $K^* := \{y \in V : \langle y, x \rangle \geq 0\ \forall\, x \in K\}$.

L_A and S_A have the **z**-property on the semidefinite cone.

Properties equivalent to the **Z**-property

Schneider-Vidyasagar (1970)

- $exp(-tL)(K) \subseteq K$ for all $t \geq 0$.
- $\dot{x} + L(x) = 0,\ x(0) \in K \Rightarrow x(t) \in K$ for all $t \geq 0$.

Stern (1981):

For a **Z**-transformation, the following are equivalent:

- There exists $d \in int(K)$ such that $L(d) \in int(K)$.
- L^{-1} exists and $L^{-1}(K) \subseteq K$.
- L is positive stable.

Gowda-Tao (2009): The above are further equivalent to:

L has the **Q**-property with respect to K.

Open problem: Does L have the **P**-property?

Switched linear systems

$$\{A_1, A_2, \ldots, A_m\} \subset R^{n\times n}.$$

$$\dot{x} + A_\sigma x = 0, \quad \sigma \in \{1, 2, \ldots, m\}$$

is a switched linear system.

Switching Signal $\sigma : [0, \infty) \to \{1, 2, \ldots, m\}$

is piecewise constant.

Survey article by R. Shorten et al in SIAM Review, 2007,

Monograph: Switching systems by D. Liberzon.

The (uniform exponential) asymptotic stability:

There exists $\beta > 0$ such that

$$||x(t)|| \leq Me^{-\beta t}||x(0)||$$

for all t, all solutions, and all signals.

Sufficient condition for asymptotic stability:

$$P \succ 0 \quad \text{and} \quad A_i^T P + P A_i \succ 0 \text{ for all } i = 1, 2, \ldots, m.$$

Then $V(x) := x^T P x$ is a **Common Quadratic Lyapunov Function.**

Question: When do we have a CQLF?

Known conditions: Commutativity, simultaneous triangularization, Lie algebraic conditions, etc.

Similar results/ideas for discrete switched systems.

Kamenetskiy and Pyatnitskiy (1987):

There exists a CQLF if and only if

$$\sum_{i=1}^{m}(A_iY_i + Y_iA_i^T) = 0,\ \ Y_i \succeq 0, \forall i \Rightarrow Y_i = 0\ \forall\, i.$$

$L_i : H \to H$ linear for $i = 1, 2, \ldots, m$.

L_1 is positive stable and has the **z**-property on proper cone K.

Moldovan-Gowda (2009) The following are equivalent:

- There exists $d \in K^\circ$ such that $L_i(d) \in K^\circ$ for all $i = 1, ..., m$.
- $\sum_1^m L_i^T(y_i) = 0,\ y_i \in K^*\ \forall i \ \Rightarrow y_i = 0\, \forall i$.

Proof based on duality/Theorem of alternative:

Let $K_1 \subseteq H_1$ and $K_2 \subseteq H_2$.

$L : H_1 \to H_2$ linear.

Then exactly one of the following is consistent:

- $L(x) \in (K_2)^\circ, x \in (K_1)^\circ$.
- $L^T(y) \in K_1^*, 0 \neq y \in -K_2^*$.

Positive switched systems

$\dot{x} + A_\sigma x = 0, \quad \sigma \in \{1, 2, \ldots, m\}$

where each A_i is a **z**-matrix.

$Q(x) = x^T d$ is a common linear Lyapunov function

if $d > 0$ and $A_i^T d > 0$ for all i.

In this case, the switched system is asymptotically stable.

Question: When do we have such a d?

Complementarity ideas again

$$x \wedge y = \min\{x, y\}$$

Vertical linear complementarity problem:

$$x \wedge (A_1^T x + q_1) \wedge \cdots \wedge (A_m^T x + q_m) = 0.$$

If $q_i < 0$, then $x \geq 0$ and $A_i^T x \geq -q_i > 0$ for all i;

such an x produces a vector $d > 0$ such that

$A_i^T d > 0$ for all i.

Gowda-Sznajder (1994)

VLCP has a unique solution for all $q_1, q_2, \cdots, q_m$

if and only if all column representatives of

$\mathcal{A} = \{A_1, A_2, ..., A_m\}$ are **P**-matrices.

Mason-Shorten (2007)

A_1 and A_2 are positive stable $\mathbf{Z}$-matrices.

The following are equivalent:

- There exists $d > 0$ in R^n such that $A_1^T d > 0$ and $A_2^T d > 0$.
- Every column rep. of $\{A_1, A_2\}$ is positive stable.
- Every column rep. of $\{A_1, A_2\}$ has positive determinant.

Song-Gowda-Ravindran (2003)

$\mathcal{A}$ is a compact set of positive stable **z**-matrices:

- There exists $d > 0$ such that $A^T d > 0$ for all $A \in \mathcal{A}$.
- Every column rep. of $\mathcal{A}$ is a $\mathbf{P}$-matrix.

$\mathcal{A} = \{A_1, A_2, ..., A_m\}$ pos. stable $\mathbf{Z}$-matrices

$\mathcal{A}^{\#}$:= set of all column representatives of $\mathcal{A}$

$$\widehat{\mathcal{A}} := \left\{ \sum_{1}^{m} A_i D_i \;:\; D_i \text{ is nonnegative and diagonal}, \; \sum_{1}^{m} D_i = I \right\}$$

Moldovan- Gowda (2009) The following are equivalent:

(1) There exists $d > 0$ such that $A_i^T d > 0$ for all $i = 1, 2, \ldots, m$.

(2) Every matrix in $\widehat{\mathcal{A}}$ is a $\mathbf{P}$-matrix.

(3) Every matrix in $\mathcal{A}^{\#}$ is a $\mathbf{P}$-matrix.

(4) Every matrix in $\mathcal{A}^{\#}$ has positive determinant.

(5) Every matrix in $\widehat{\mathcal{A}}$ has positive determinant.

(6) Every VLCP corresponding to $\mathcal{A}^{\mathcal{T}}$ has a unique solution.

Let $\mathcal{A}$ be a compact set of positive stable $\mathbf{Z}$-matrices in $R^{n \times n}$. Then the following are equivalent:

(1) There exists a $d > 0$ such that $A^T d > 0$ for all $A \in \mathcal{A}$.

(2) Every column representative of $\mathcal{A}$ is a $\mathbf{P}$-matrix.

(3) Every column representative of $\mathcal{A}$ has positive determinant.