$\mathbf{Z}$ matrices, linear transformations, and tensors

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This is an expository talk on $\mathbb{Z}$ matrices, transformations on proper cones, and tensors. The objective is to show that these have very similar properties.
Outline

- The $Z$-property
- $M$ and strong (nonsingular) $M$-properties
- The $P$-property
- Complementarity problems
- Zero-sum games
- Dynamical systems
Some notation

- $\mathbb{R}^n$: The Euclidean $n$-space of column vectors.
- $\mathbb{R}^n_+$: Nonnegative orthant, $x \in \mathbb{R}^n_+ \iff x \geq 0$.
- $\mathbb{R}^{n_+}_+$: The interior of $\mathbb{R}^n_+$, $x \in^{n_+}_+ \iff x > 0$.
- $\langle x, y \rangle$: Usual inner product between $x$ and $y$.
- $\mathbb{R}^{n \times n}$: The space of all $n \times n$ real matrices.
- $\sigma(A)$: The set of all eigenvalues of $A \in \mathbb{R}^{n \times n}$. 
The $Z$-property

$A = [a_{ij}]$ is an $n \times n$ real matrix

- $A$ is a $Z$-matrix if $a_{ij} \leq 0$ for all $i \neq j$.

  (In economics literature, $-A$ is a Metzler matrix.)

- We can write $A = rI - B$, where $r \in \mathbb{R}$ and $B \geq 0$.

  Let $\rho(B)$ denote the spectral radius of $B$.

- $A$ is an $M$-matrix if $r \geq \rho(B)$,

- nonsingular (strong) $M$-matrix if $r > \rho(B)$.  

The $P$-property

- $A$ is a $P$-matrix if all its principal minors are positive.

**Theorem:** The following are equivalent:

- $A$ is a $P$-matrix.
- $x \ast Ax \leq 0 \Rightarrow x = 0$.
- $\max_i x_i(Ax)_i > 0$ for all $x \neq 0$.

Here, $x \ast y$ denotes the componentwise product of vectors $x$ and $y$. 
**Theorem:** The following are equivalent for a $Z$-matrix $A$:

- $A$ is a $P$-matrix.
- $A$ is a nonsingular $M$-matrix.
- There exists a vector $d > 0$ such that $Ad > 0$.
- $A$ is invertible and $A^{-1}$ is a nonnegative matrix.

The book on Nonnegative matrices by Berman and Plemmons lists 52 equivalent conditions.
The linear complementarity problem

Given $A \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, $\text{LCP}(A, q)$ :

Find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \ y := Ax + q \geq 0, \ \text{and} \ \langle x, y \rangle = 0.$$ 

Theorem:

- $A$ is a $P$-matrix iff $\text{LCP}(A, q)$ has a unique solution for all $q$.
- Suppose $A$ is a $Z$-matrix. If $\{x \geq 0, Ax + q \geq 0\}$ is nonempty, then its least element solves $\text{LCP}(A, q)$. 

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Two players $I$ and $II$ start with a matrix $A$ and

**Strategy set** $\Delta := \{ x \in R^n_+ : \sum_1^n x_i = 1 \}$.

$I$ chooses $x \in \Delta$ and $II$ chooses $y \in \Delta$.

Then, payoff for $I$ is $\langle Ax, y \rangle$ and payoff for $II$ is $-\langle Ax, y \rangle$.

**Theorem:** There exist ‘optimal strategies’ $p, q \in \Delta$ such that

$$\langle Ax, q \rangle \leq \langle Ap, q \rangle \leq \langle Ap, y \rangle \quad \text{for all } x, y \in \Delta.$$ 

The value of the game $v(A) := \langle Ap, q \rangle$.

$$v(A) := \max_{x \in \Delta} \min_{y \in \Delta} \langle Ax, y \rangle = \min_{y \in \Delta} \max_{x \in \Delta} \langle Ax, y \rangle.$$
The game is *completely mixed* if $p > 0$ and $q > 0$ for all optimal strategy pairs.

**Theorem:** (Kaplansky, 1945) *If the game is completely mixed, then the game has unique optimal strategies.*

**Theorem:** (Raghavan, 1978) *Suppose $A$ is a $Z$-matrix. Then, $v(A) > 0$ iff $A$ is a $P$-matrix. In this case, the game is completely mixed.*
Dynamical systems

Given $A$, consider the continuous linear dynamical system

$$\frac{dx}{dt} + Ax = 0, \ x(0) \in \mathbb{R}^n.$$ 

Its trajectory in $\mathbb{R}^n$: $x(t) = e^{-tA}x(0)$ for all $t \in \mathbb{R}$.

**Theorem:** $A$ is a Z-matrix iff $e^{-tA}(\mathbb{R}^n_+) \subseteq \mathbb{R}^n_+$ for all $t \geq 0$. 
Theorem: Suppose $A$ is a $Z$-matrix. Then, the following are equivalent:

- There exists $d > 0$ such that $Ad > 0$.
- $A$ is a $P$-matrix.
- $A$ is positive stable: All eigenvalues of $A$ have positive real parts.
- For any $x(0) \in \mathbb{R}^n$, $x(t) \to 0$ as $t \to \infty$. 
Proper cones

- $H$: Finite dimensional real inner product space.
- $\langle x, y \rangle$: inner product between $x, y \in H$.
- $K$: Proper cone in $H$.
  (closed convex pointed cone with nonempty interior)
- The dual cone $K^* := \{ x \in H : \langle x, y \rangle \geq 0 \ \forall \ y \in K \}$. 
$Z$ and Lyapunov-like transformations

$K$ is a proper cone in $H$, $L : H \to H$ is linear.

- $L$ is a $Z$-transformation on $K$ if
  
  $x \in K, \ y \in K^*, \ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0.$

  We write $L \in Z(K)$.

(Schneider-Vidyasagar: $L$ is cross-positive if $-L \in Z(K)$.)

- $L$ is a Lyapunov-like transformation on $K$ if
  
  $x \in K, \ y \in K^*, \ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle = 0.$

  We write $L \in LL(K)$. 
Examples

• \( H = S^n \): Space of all \( n \times n \) real symmetric matrices.
• \( \langle X, Y \rangle := \text{trace}(XY) \).
• \( K = S^n_+ \): cone of positive semidefinite matrices in \( S^n \).
• For any \( A \in \mathbb{R}^{n \times n} \),
  \[
  S_A(X) := X - AXA^T \quad \text{and} \quad L_A(X) := AX + XA^T.
  \]
  (\( S_A \)-Stein transformation, \( L_A \)-Lyapunov transformation)
• \( S_A \in Z(S^n_+) \) and \( L_A \in LL(S^n_+) \).
Properties equivalent to the $Z$-property on a proper cone $K$:

(Schneider-Vidyasagar, 1970)

- $e^{-tL}(K) \subseteq K$ for all $t \geq 0$.
- $L = \lim_{n \to \infty} (t_n I - S_n)$ where $S_n(K) \subseteq K$ and $t_n \in R$.

Properties equivalent to the $LL$-property on a proper cone $K$:

- $e^{tL}(K) \subseteq K$ for all $t \in R$.
- $L \in \text{Lie} (\text{Aut}(K))$ (Lie algebra characterization).

Lyapunov rank of $K := \text{dimension of Lie}(\text{Aut}(K))$.

Useful in conic optimization.
Lyapunov rank

If the Lyapunov rank of $K$ is more than the dimension $n$ of the ambient space, then the complementarity system

$$x \in K, \ y \in K^*, \ \langle x, y \rangle = 0$$

can be expressed in terms of $n$ linearly independent bilinear relations.

For example, in $\mathbb{R}^n$, when $x \geq 0, \ y \geq 0$,

$$\langle x, y \rangle = 0 \iff x_i y_i = 0 \ \forall \ i.$$

This idea is useful in linear programs over cones.
Theorem: (Stern 1981, Gowda-Tao 2006)
The following are equivalent for a $Z$-transformation:

- There exists $d \in \text{int}(K)$ such that $L(d) \in \text{int}(K)$.
- $L^{-1}$ exists and $L^{-1}(K) \subseteq K$.
- $L$ is positive stable.
- All real eigenvalues of $L$ are positive.
- For any $q \in H$, there exists $x$ such that $x \in K$, $L(x) + q \in K^*$, $\langle x, L(x) + q \rangle = 0$.

Recall: $K^* := \{y \in H : \langle y, x \rangle \geq 0 \ \forall x \in K\}$. 
Euclidean Jordan algebras

For a matrix $A \in \mathbb{R}^{n \times n}$, the following are equivalent:

- $A$ is a $P$-matrix: $x \ast A x \leq 0 \Rightarrow x = 0$.
- All principal minors of $A$ are positive.

How to generalize the $P$-property?

$(V, \langle \cdot, \cdot \rangle, \circ)$ is a Euclidean Jordan algebra if $V$ is a finite dimensional real inner product space and the bilinear Jordan product $x \circ y$ satisfies:

- $x \circ y = y \circ x$
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$
Examples

Any EJA is a product of the following simple algebras:

- $S^n = \text{Herm}(\mathcal{R}^{n \times n})$: All $n \times n$ real symmetric matrices.
- $\text{Herm}(\mathcal{C}^{n \times n})$: $n \times n$ complex Hermitian matrices.
- $\text{Herm}(\mathcal{Q}^{n \times n})$: $n \times n$ quaternion Hermitian matrices.
- $\text{Herm}(\mathcal{O}^{3 \times 3})$: $3 \times 3$ octonion Hermitian matrices.
- $\mathcal{L}^n (n \geq 3)$: The Jordan spin algebra.

$K = \{ x^2 : x \in V \}$ is the symmetric cone of $V$.

It is a self-dual, homogeneous cone.
Characterization of LL transformations on a EJA:

**Theorem:** (Tao-Gowda, 2013) The following are equivalent:

- $L$ is Lyapunov-like on the symmetric cone of $V$,
- $L = L_a + D$, where $a \in V$ and $D : V \to V$ is a derivation.

Here,

$L_a(x) := a \circ x$ and $D(x \circ y) = D(x) \circ y + x \circ D(y)$.

**Corollary:** (Damm, 2004) On $S^n$, $L$ is Lyapunov-like iff it is of the form $L_A$ for some $A \in R^{n \times n}$.
The $P$-property

On an EJA $V$, $a$ and $b$ operator commute if

$$L_a L_b = L_b L_a.$$ 

A linear $L : V \to V$ has the $P$-property if

$x$ and $L(x)$ operator commute, $x \circ L(x) \leq 0 \Rightarrow x = 0$.

**Theorem:** (Gowda, Tao, and Ravindran, 2012)

Suppose $L$ is Lyapunov-like on the symmetric cone of $V$.

$L$ has $P$-property if and only if

- There exists $d \in \text{int}(K)$ such that $L(d) \in \text{int}(K)$.

**Conjecture:** This result holds for $Z$-transformations.
Lyapunov’s theorem revisited

Recall: $L_A(X) := AX + XA^T$ on $S^n$.

**Corollary:** For any $A \in \mathbb{R}^{n \times n}$, the following are equivalent:

- $L_A$ has $P$-property on $S^n_+$. 
- There exists $X \succ 0$ such that $AX + XA^T \succ 0$.
- $A$ is positive stable.

Here, $X \succ 0$ means that $X$ is symmetric and positive definite.
Let $K$ be *self-dual* that is, $K = K^*$ in $H$. For $e \in \text{int}(K)$, define

$$\Delta := \{ x \in K : \langle x, e \rangle = 1 \}.$$  

(We think of $\Delta$ as the strategy set.)

Given a linear transformation $L : H \to H$, by von Neumann’s min-max theorem, there exist

‘optimal strategies’ $p$ and $q$ such that

$$\langle L(x), q \rangle \leq \langle L(p), q \rangle \leq \langle L(p), y \rangle$$

for all strategies $x, y \in \Delta$.

$v(L) := \langle L(p), q \rangle$ is called the value of $L$. 

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A generalization of Raghavan’s result:

**Theorem:** (Gowda-Ravindran, 2015) Let $K$ be self-dual and $L \in Z(K)$. Then the following are equivalent:

- $v(L) > 0$.
- There exists $d \in \text{int}(K)$ such that $L(d) \in \text{int}(K)$.
- $L$ is positive stable.

In this case, $L$ is completely mixed.

Moreover, if $L$ is Lyapunov-like and $v(L) \neq 0$, then

$L$ is completely mixed.
A tensor is a multidimensional analog of a matrix.

Let \( \mathcal{A} := [a_{i_1i_2\ldots i_m}] \) be an \( m \)-th order, \( n \)-dimensional tensor.

The entries \( a_{i_1i_2\ldots i_m} \), \( 1 \leq i \leq n \), are the diagonal entries of \( \mathcal{A} \) and the rest are ‘off-diagonal’ entries.

A nonnegative tensor has all entries nonnegative.

A tensor is a \( \mathcal{Z} \)-tensor if all its off-diagonal entries are non-positive.

Such a tensor can be written as \( \mathcal{A} = r\mathcal{I} - \mathcal{B} \), where \( r \in \mathbb{R} \), \( \mathcal{I} \) is the identity tensor, and \( \mathcal{B} \) is a nonnegative tensor.
Given a tensor $\mathcal{A} := [a_{i_1 i_2 \cdots i_m}]$, we define the function

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ whose } i\text{th component is given by}$$

$$F_i(x) = \sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i_2 i_3 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}.$$ 

Commonly used notation: $\mathcal{A} x^{m-1} := F(x)$.

Each component of $F(x)$ is a homogeneous polynomial of degree $m - 1$. 
Given a tensor $\mathcal{A}$,

- $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ if $\exists$ nonzero $x \in \mathbb{C}^n$ such that $\mathcal{A}x^{m-1} = \lambda x^{m-1}$.

$$\sigma(\mathcal{A}) := \text{Set of all eigenvalues of } \mathcal{A}.$$  

- Spectral radius $\rho(\mathcal{A}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$.

**Theorem:** (Yang-Yang, 2010)

For a nonnegative tensor, its spectral radius is an eigenvalue with a nonnegative eigenvector.
Given a tensor $\mathcal{A}$ and $q \in \mathbb{R}^n$, the

- **Tensor complementarity problem** $\text{TCP}(\mathcal{A}, q)$ is to find $x \in \mathbb{R}^n$ such that

  $x \geq 0$, $y = F(x) + q \geq 0$, and $\langle x, y \rangle = 0$.

Note: $F(x) = \mathcal{A}x^{m-1}$ is a polynomial map.

Thus, $\text{TCP}$ is a *nonlinear complementarity problem*.

Equation formulation: $\min\{x, F(x) + q\} = 0$. 
Theorem: (Luo, Qi, Xiu, 2015)

Suppose $A$ is a $Z$-tensor. If $TCP(A, q)$ is feasible, then it is solvable.
A Mega theorem

Theorem: $A = rI - B$ with $B$ nonnegative, $F(x) = Ax^{m-1}$.

Then the following are equivalent:

- There exists $d > 0$ such that $F(d) > 0$.
- $A$ is positive stable: $Re(\lambda) > 0 \ \forall \ \lambda \in \sigma(A)$.
- $A$ is a strong $M$-tensor: $r > \rho(B)$.
- $A$ is a $P$-tensor: $\forall x \neq 0$, $\max_i x_i^{m-1}(Ax^{m-1})_i > 0$.
- $TCP(A, q)$ is solvable for all $q$.
- $TCP(A, q)$ has trivial solution for all $q \geq 0$.
- $\mathbb{R}^n_+ \subseteq F(\mathbb{R}^n_+)$.
- For all $x \geq 0$, $\max_i x_i(Ax^{m-1})_i > 0$. 
Mega theorem, continued

If $A$ is of even order, we have further equivalence:

- $F$ is surjective.
- For all $x \neq 0$, $\max_i x_i(Ax^{m-1})_i > 0$.

Open problem: Characterize $Z$-tensors $A$ for which $TCP(A, q)$ has a unique solution for all $q$. 
A viability result

Suppose \( \mathcal{A} \) is a \( \mathcal{Z} \)-tensor.

Then, \(-F(x)\) is ‘cooperative’ (that is, its derivative is a Metzler matrix at all points).

**Theorem:** Let \( x(t) \) solve the dynamical system

\[
\frac{dx}{dt} + F(x) = 0, \quad x(0) = x_0.
\]

If \( x(0) \in \mathbb{R}^n_+ \), then \( x(t) \in \mathbb{R}^n_+ \) for all \( t \).

This comes from an application of Nagumo’s theorem on viability (also called Bony-Brezis theorem).
Asymptotic convergence

**Theorem:** Suppose $\mathcal{A}$ is a $\mathbb{Z}$-tensor.

Then, $\frac{dx}{dt} + F(x) = 0$ is exponentially stable iff $\mathcal{A}$ is a strong $M$-tensor.

Tensor zero-sum games

No results yet......