On the game-theoretic value of a linear transformation on a symmetric cone

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Consider two players I and II with payoff matrix $A \in \mathbb{R}^{n \times n}$. Player I chooses columns of $A$ with probability/strategy $x$: $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, $x \geq 0$, $\sum_1^n x_i = 1$; Player II chooses rows of $A$ with probability/strategy $y$. Then payoff for I is $\langle Ax, y \rangle$ and payoff for II is $-\langle Ax, y \rangle$.

**Theorem of von Neumann**: There exist optimal/equilibrium strategies $\bar{x}$ and $\bar{y}$ such that

$$\langle Ax, \bar{y} \rangle \leq \langle A\bar{x}, \bar{y} \rangle \leq \langle A\bar{x}, y \rangle$$

for all strategies $x$ and $y$. 
\( v(A) := \langle A\bar{x}, \bar{y} \rangle \) is called the value of the game.

**Min-max Theorem of von Neumann:**

\[
\max_{x \in \Delta} \min_{y \in \Delta} \langle Ax, y \rangle = \min_{y \in \Delta} \max_{x \in \Delta} \langle Ax, y \rangle,
\]

where \( \Delta := \{ x = (x_1, x_2, \ldots, x_n)^T \in R^n : x \geq 0, \sum_1^n x_i = 1 \} \).

The common value is \( v(A) \).
A zero-sum matrix game is a special case of Bimatrix game.

A bimatrix game is a special case of (standard) linear complementarity problem.
Uniqueness of optimal strategies

A game is completely mixed if $\bar{x} > 0$ and $\bar{y} > 0$ for every optimal pair $(\bar{x}, \bar{y})$.

Theorem of Kaplansky (1945):

(i) If $\bar{y} > 0$ for every optimal pair $(\bar{x}, \bar{y})$, then the game is completely mixed.

(ii) If the game is completely mixed, then the optimal pair is unique.
A square real matrix $A = [a_{ij}]$ is a $Z$-matrix if $a_{ij} \leq 0$ for all $i \neq j$.

(In Economics, $-A$ is called a Metzler matrix.)

**Theorem of Raghavan (1978):**

Let $A$ be a $Z$-matrix.

(i) If $v(A) > 0$, then the game is completely mixed.

(ii) $v(A) > 0$ iff there exists $d > 0$ in $\mathbb{R}^n$ such that $Ad > 0$.

Note: For a $Z$-matrix, the condition $d > 0$, $Ad > 0$ can be described in more than 52 equivalent ways.
Dynamical systems

For an $n \times n$ real matrix $A$, the continuous dynamical system $\frac{dx}{dt} + Ax(t) = 0$ is asymptotically stable on $\mathbb{R}^n$ (i.e., any trajectory starting from an arbitrary point in $\mathbb{R}^n$ converges to the origin) if and only if there exists a real symmetric matrix $D$ such that

$$D \succ 0 \quad \text{and} \quad L_A(D) \succ 0,$$

where $D \succ 0$ means that $D$ is positive definite, and

$$L_A(X) := AX + XA^T \quad (X \in \mathcal{S}^n).$$
Here, $S^n$ denotes the space of all $n \times n$ real symmetric matrices and $L_A$ is the so-called Lyapunov transformation. Similarly, the discrete dynamical system $x(k + 1) = Ax(k)$, $k = 0, 1, \ldots$, is asymptotically stable on $R^n$ if and only if there exists a real symmetric matrix $D$ such that

$$D \succ 0 \quad \text{and} \quad S_A(D) \succ 0,$$

where $S_A$ is the so-called Stein transformation on $S^n$:

$$S_A(X) := X - AXA^T \quad (X \in S^n).$$
Note similar inequalities:

- In $\mathbb{R}^n$, $d > 0$, $Ad > 0$ for a $\mathbf{Z}$-matrix
- In $\mathbb{S}^n$, $D \succ 0$ and $L_A(D) \succ 0$
- In $\mathbb{S}^n$, $D \succ 0$ and $S_A(D) \succ 0$

Why is this happening? Is there a unifying result?
• $\mathbb{R}^n$ and $\mathbb{S}^n$ are both Euclidean Jordan algebras,

• In $\mathbb{R}^n$, let $e = (1, 1, \ldots, 1)$. Then $x = (x_1, x_2, \ldots, x_n)$ is a probability vector iff all its eigenvalues $x_1, x_2, \ldots, x_n$ are nonnegative and $\langle x, e \rangle = \sum_1^n x_i = 1$.

In $\mathbb{S}^n$, let $e = I$ (Identity matrix). We could consider $X \in \mathbb{S}^n$ with all its eigenvalues nonnegative and $\langle X, I \rangle = \text{trace}(X) = \sum_1^n \lambda_i(X) = 1$. 
Z-transformations

• In $\mathbb{R}^n$, let $K = \mathbb{R}^n_+$. Then $A \in \mathbb{R}^{n \times n}$ is a $z$-matrix iff

$$x \in K, y \in K^*(= K), \text{ and } \langle x, y \rangle = 0 \Rightarrow \langle Ax, y \rangle \leq 0.$$ 

• In $\mathbb{S}^n$, let $K = \mathbb{S}^n_+$ (symmetric cone of $\mathbb{S}^n$). Then, for any $A \in \mathbb{R}^{n \times n}$,

$$X \in K, Y \in K^*(= K), \text{ and } \langle X, Y \rangle = 0 \Rightarrow \langle L_A(X), Y \rangle = 0.$$ 

• In $\mathbb{S}^n$, let $K = \mathbb{S}^n_+$. Then, for any $A \in \mathbb{R}^{n \times n}$,

$$X \in K, Y \in K^*(= K), \text{ and } \langle X, Y \rangle = 0 \Rightarrow \langle S_A(X), Y \rangle \leq 0.$$
Euclidean Jordan algebras

A Euclidean Jordan algebra is a triple $(V, \circ, \langle \cdot, \cdot \rangle)$, where $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional real inner product space and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(i) $x \circ y = y \circ x$, $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, and

(ii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$.

Examples: The Jordan spin algebra $\mathcal{L}^n$, the algebra(s) of $n \times n$ real/complex/quaternion Hermitian matrices, and the algebra of $3 \times 3$ octonion Hermitian matrices. Any nonzero EJA is a product of these.
Let $V$ be a EJA and $K$ be its symmetric cone. A linear transformation $L : V \rightarrow V$ is

- a **$Z$-transformation** if

  $$x \in K, y \in K^* (= K), \quad \text{and} \quad \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0,$$

- **Lyapunov-like** if

  $$x \in K, y \in K^* (= K), \quad \text{and} \quad \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle = 0.$$

**Theorem:** On a EJA $V$, $L$ is Lyapunov-like iff

$$L = L_a + D,$$

where $a \in V$, $L_a(x) := a \circ x$ and $D$ is a derivation.
How to define the value on a EJA?
Can we extend the results of Kaplansky and Raghavan?

From now on, $V$ is an EJA with its symmetric cone $K$.

$e$ is the unit element in $V$ and

$\Delta := \{ x \in K : \langle x, e \rangle = 1 \}$.

We think of $\Delta$ as the strategy set. Also, instead of the unit element, we could take any $e \in \text{int}(K)$.)
Given a linear transformation $L : V \rightarrow V$, as $\Delta$ is compact convex, by a Theorem of von Neumann, there exist optimal strategies $\bar{x}$ and $\bar{y}$ such that

$$\langle L(x), \bar{y} \rangle \leq \langle L(\bar{x}), \bar{y} \rangle \leq \langle L(\bar{x}), y \rangle$$

for all strategies $x, y \in \Delta$.

$v(L) := \langle L(\bar{x}), \bar{y} \rangle$ is called the value of $L$.

We have

$$v(L) = \max_{x \in \Delta} \min_{y \in \Delta} \langle L(x), y \rangle = \min_{y \in \Delta} \max_{x \in \Delta} \langle L(x), y \rangle.$$
We write \( x \geq 0 \) when \( x \in K \) and \( x > 0 \) when \( x \in \text{int}(K) \).

Let \( L \) be linear on \( V \).

**Theorem:**

(i) \((\bar{x}, \bar{y})\) is an optimal pair iff \( L^T(\bar{y}) \leq v e \leq L(\bar{x}) \).

(ii) If \((\bar{x}, \bar{y})\) is an optimal pair, then

\[
0 \leq \bar{x} \perp v e - L^T(\bar{y}) \geq 0 \quad \text{and} \quad 0 \leq \bar{y} \perp L(\bar{x}) - v e \geq 0,
\]

In addition, \( \bar{x} \) and \( L^T(\bar{y}) \) operator commute and \( \bar{y} \) and \( L(\bar{x}) \) operator commute.
Corollary:

(i) \( v(-L^T) = -v(L) \).

(ii) \( v(L + \lambda e e^T) = v(L) + \lambda \).

(iii) For any \( A \in Aut(K) \), and \( e \in int(K) \),

\[
v(ALA^T, Ae) = v(L, e).
\]

Note that as \( K \) is homogeneous, one could go from one interior point of \( K \) to another. Thus, for many properties, chosen interior point \( e \) is unimportant.
We say that $L$ is *completely mixed* if
\[ \bar{x} > 0 \text{ and } \bar{y} > 0 \text{ for every optimal pair } (\bar{x}, \bar{y}). \]

An EJA generalization of Kaplansky’s Theorem:

(i) If $\bar{y} > 0$ for every optimal pair $(\bar{x}, \bar{y})$,
then $L$ is completely mixed.

(ii) If $L$ is completely mixed, then the optimal pair is unique.

(iii) If $L$ is completely mixed, then so is $L^T$ and
\[ v(L) = v(L^T). \]
An EJA generalization of Raghavan’s Theorem:

The following are equivalent when \( L \) is a \( Z \)-transformation:

(i) \( v(L) > 0 \).

(ii) \( L \) is positive stable (real parts of eigenvalues of \( L \) are positive).

(iii) There exists \( d > 0 \) such that \( L(d) > 0 \).

In addition, \( L \) is completely mixed when \( v(L) > 0 \).

What happens when \( v(L) < 0 \)?
Easy examples show that a $\mathbf{Z}$-transformation $L$ may not be completely mixed when $v(L) < 0$. However, we have

**Theorem:**

When $L$ is Lyapunov-like (or Stein-like*) and $v(L) \neq 0$, $L$ is completely mixed.

$L$ is said to be **Stein-like** if $L = I - \Lambda$, where $\Lambda \in \text{Aut}(K)$.

Example: The Stein transformation $S_A$ on $S^n$ given by

$$S_A(X) = X - AXA^T.$$
Value inequalities

Let $V = V_1 \times V_2$ with $K = K_1 \times K_2$.

Let $L$ be linear on $V$. We write

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A : V_1 \to V_1$ is linear, etc. If $A$ is invertible, we define the Schur complement

$$L/A := D - CA^{-1}B.$$
Theorem: Suppose $L$ is a $\mathbf{Z}$-transformation. Then $A$ is $\mathbf{Z}$ on $V_1$ and $D$ is $\mathbf{Z}$ on $V_2$.

(i) If $v(L) > 0$, then $v(A) > 0$, $v(D) > 0$ and

$$\frac{1}{v(L)} \geq \frac{1}{v(A)} + \frac{1}{v(D)}.$$

Reverse implications and inequalities hold if $L$ is Lypaunov-like.

(ii) If $v(L) > 0$, then $v(A) > 0$, $v(L/A) > 0$ and

$$\frac{1}{v(L)} \geq \frac{1}{v(A)} + \frac{1}{v(L/A)}.$$
Concluding remarks

(i) The value of a linear transformation on a EJA can be computed by a (symmetric) cone linear program in polynomial time.

(ii) Let $L$ be completely mixed. If $L$ is invertible, then, $v(L) = \frac{1}{\langle L^{-1}(e), e \rangle}$. Also, the unique optimal pair is given by $\bar{x} = v(L) L^{-1}(e)$ and $\bar{y} = v(L) (L^T)^{-1}(e)$.

(iii) Many of the results presented here carry over to self-dual cones.
Some references

(1) Berman and Plemmons, Nonnegative matrices in Mathematical Sciences.

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(3) Kaplansky, A contribution to von Neumann’s theory of games.

(4) Karlin, Mathematical methods and theory in games.

(5) Parthasarathy and Raghavan, Some topics in two-person games.

(6) Raghavan, Completely mixed games and M-matrices.