On the connectedness of spectral sets and irreducibility of spectral cones in Euclidean Jordan algebras

M. Seetharama Gowda*, Juyoung Jeong

Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, MD 21250, USA

A R T I C L E   I N F O

Article history:
Received 4 May 2018
Accepted 5 September 2018
Available online 11 September 2018
Submitted by R. Brualdi

MSC:
54D05
17C20
17C30
22E99
15B48

Keywords:
Euclidean Jordan algebra
Spectral set
Connectedness
Irreducible cone

A B S T R A C T

Let \( V \) be a Euclidean Jordan algebra of rank \( n \). A set \( E \) in \( V \) is said to be a spectral set if there exists a permutation invariant set \( Q \) in \( \mathbb{R}^n \) such that \( E = \lambda^{-1}(Q) \), where \( \lambda : V \to \mathbb{R}^n \) is the eigenvalue map that takes \( x \in V \) to \( \lambda(x) \) (the vector of eigenvalues of \( x \) written in the decreasing order). If the above \( Q \) is also a convex cone, we say that \( E \) is a spectral cone. This paper deals with connectedness and arcwise connectedness properties of spectral sets. By relying on the result that in a simple Euclidean Jordan algebra, every eigenvalue orbit \( [x] := \{ y : \lambda(y) = \lambda(x) \} \) is arcwise connected, we show that if a permutation invariant set \( Q \) is connected (arcwise connected), then \( \lambda^{-1}(Q) \) is connected (respectively, arcwise connected). A related result is that in a simple Euclidean Jordan algebra, every pointed spectral cone is irreducible.

© 2018 Elsevier Inc. All rights reserved.

* Corresponding author.
E-mail addresses: gowda@umbc.edu (M.S. Gowda), juyoung1@umbc.edu (J. Jeong).

https://doi.org/10.1016/j.laa.2018.09.006
0024-3795/© 2018 Elsevier Inc. All rights reserved.
1. Introduction

Let $V$ be a Euclidean Jordan algebra of rank $n$ [8]. The eigenvalue map $\lambda : V \to \mathcal{R}^n$ takes $x \in V$ to $\lambda(x)$, the vector of eigenvalues of $x$ written in the decreasing order. A set $E$ in $V$ is said to be a spectral set if there exists a permutation invariant set $Q$ in $\mathcal{R}^n$ such that $E = \lambda^{-1}(Q)$. If the above $Q$ is also a convex cone, we say that $E$ is a spectral cone. A function $F : V \to \mathcal{R}$ is said to be a spectral function if it is of the form $F = f \circ \lambda$, where $f : \mathcal{R}^n \to \mathcal{R}$ is a permutation invariant function. The above concepts are generalizations of similar concepts that have been studied in the settings of Euclidean $n$-space $\mathcal{R}^n$ and $S^n (\mathcal{H}^n)$, the space of all $n \times n$ real (respectively, complex) Hermitian matrices, see for example, [2], [3], [4], [5], [12], [13], [17], [18], [19], and [22], and the references therein. In the case of $\mathcal{R}^n$, spectral sets/cones/functions are related to permutation invariance, and in $S^n (\mathcal{H}^n)$ they are precisely those that are invariant under linear transformations of the form $X \to UXU^*$, where $U \in \mathcal{R}^{n \times n}$ is an orthogonal (respectively, unitary) matrix. In the general setting of Euclidean Jordan algebras, they have been studied in several works, see [1], [9], [14], [15], [16], [20], [23], and [24].

Focusing on topological/convexity/linearity properties of spectral sets/cones, in two recent papers Jeong and Gowda [15], [16] show that the multivalued map $\lambda^{-1}$ from $\mathcal{R}^n$ to $V$ behaves like a linear isomorphism on certain types of permutation invariant sets. Specifically, the following results are shown (where part of the first result is due to Baes [1]):

**Proposition 1.1.** Let $Q$, $Q_1$, and $Q_2$ be permutation invariant sets in $\mathcal{R}^n$ with $Q_1$ and $Q_2$ convex. Let $\alpha \in \mathcal{R}$. Then

(a) $\lambda^{-1}(Q)$ is open/closed/compact/convex/cone in $V$ if and only if $Q$ is so in $\mathcal{R}^n$. Moreover, $\lambda^{-1}(Q^\#) = [\lambda^{-1}(Q)]^\#$, where $\#$ denotes any operation of taking closure, interior, boundary, or convex/conic hull.

(b) $\lambda^{-1}(Q_1 + Q_2) = \lambda^{-1}(Q_1) + \lambda^{-1}(Q_2)$ and $\lambda^{-1}(\alpha Q_1) = \alpha \lambda^{-1}(Q_1)$.

(c) When $Q$ is a convex cone, $\lambda^{-1}(Q)$ is pointed if and only if $Q$ is pointed.

Motivated by these results, we ask if connectedness and arcwise (= pathwise) connected properties of permutation invariant $Q$ carry over to $\lambda^{-1}(Q)$. In this paper, we answer these affirmatively by relying on the result that in a simple Euclidean Jordan algebra, for any element $x$, the eigenvalue orbit $[x] := \{ y : \lambda(y) = \lambda(x) \}$ is arcwise connected. This result will also be used to show that in any simple Euclidean Jordan algebra, every pointed spectral cone is irreducible.

2. Preliminaries

We let $\mathcal{R}^n$ denote the Euclidean $n$-space (where the vectors are regarded as column vectors or row vectors depending on the context). In $\mathcal{R}^n$, we denote the standard coor-
nate vectors by $c_1, c_2, \ldots, c_n$, where $c_i$ is the vector with one in the $i$th slot and zeros elsewhere. We use the notation $\Sigma_n$ to denote the set of all $n \times n$ permutation matrices. For any set $S$ in $\mathcal{R}^n$, we let $\Sigma_n(S) := \{\sigma(s) : \sigma \in \Sigma_n, s \in S\}$. For any vector $q \in \mathcal{R}^n$ and a set $Q \subseteq \mathcal{R}^n$, $q^\downarrow$ denotes the decreasing rearrangement of $x$ (i.e., the entries of $q^\downarrow$ satisfy $q_1^\downarrow \geq q_2^\downarrow \geq \cdots \geq q_n^\downarrow$) and $Q^\downarrow := \{q^\downarrow : q \in Q\}$. A non-empty set $Q$ in $\mathcal{R}^n$ is said to be permutation invariant if $\sigma(Q) = Q$ for all $\sigma \in \Sigma_n$.

We assume that the reader is familiar with standard topological notions/results dealing with (arcwise) connectedness, components, etc. Recall that a set $S$ (say, in a topological space) is connected if it is not the union of two nonempty separated sets (where separation takes the form $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, with the ‘overline’ indicating the closure) [21]. The set $S$ is arcwise (= pathwise) connected if any two points of $S$ can be joined by a continuous arc (= continuous image of an interval in $\mathcal{R}$) that lies inside $S$. Connected (arcwise connected) components of a set $S$ are maximal connected (respectively, arcwise connected) sets in $S$.

For basic things related to Euclidean Jordan algebras, we refer to [8] or [16] for a summary. Throughout this paper, $\mathcal{V}$ denotes a Euclidean Jordan algebra of rank $n$. Recall that a Euclidean Jordan algebra $\mathcal{V}$ is simple if it is not a direct product of nonzero Euclidean Jordan algebras (or equivalently, if it does not contain any non-trivial ideal). It is known (see [8], Prop. III.4.4) that any nonzero Euclidean Jordan algebra is, in a unique way, a direct product/sum of simple Euclidean Jordan algebras. Moreover, there are (in the isomorphic sense) only five simple algebras, two of which are: $S^n$, the algebra of $n \times n$ real symmetric matrices, and $\mathcal{H}^n$, the algebra of $n \times n$ complex Hermitian matrices. The other three are: $n \times n$ quaternion Hermitian matrices, $3 \times 3$ octonion Hermitian matrices, and the Jordan spin algebra. In $\mathcal{V}$, each element $x$ has a spectral decomposition: $x = q_1e_1 + q_2e_2 + \ldots + q_ne_n$, where $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{V}$ and the real numbers $q_1, q_2, \ldots, q_n$ are (called) the eigenvalues of $x$. Then $\lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x))$ denotes the vector of eigenvalues of $x$ written in the decreasing order. We note that

$$\lambda : \mathcal{V} \to \mathcal{R}^n \text{ is continuous ([1], Corollary 24) and } \lambda(\mathcal{V}) = (\mathcal{R}^n)^{\downarrow}.$$ 

For any $x \in \mathcal{V}$, we let

$$[x] := \{y \in \mathcal{V} : \lambda(y) = \lambda(x)\}$$

denote the ‘eigenvalue orbit’ of $x$. (The notation $[x]_\mathcal{V}$ will be used when more than one algebra is involved.) Suppose $\mathcal{V}$ is a Cartesian product (or a direct sum) $\mathcal{V} = \mathcal{V}^{(1)} \times \mathcal{V}^{(2)} \times \cdots \times \mathcal{V}^{(N)}$, where each $\mathcal{V}^{(i)}$ is simple. It is easy to see that in $\mathcal{V}$, an element $v$ is a primitive idempotent if and only if it is of the form $(0, 0, \ldots, 0, v^{(i)}0, \ldots, 0)$, where $v^{(i)}$ is a primitive idempotent in $\mathcal{V}^{(i)}$ for some $i$. Applying this to elements of a Jordan frame in $\mathcal{V}$, we see that the eigenvalues of any $x = (x^{(1)}, x^{(2)}, \ldots, x^{(N)}) \in \mathcal{V}$ comprise of the eigenvalues of $x^{(i)}$ in $\mathcal{V}^{(i)}$, $i = 1, 2, \ldots, N$. For such an $x$, we define the ‘restricted eigenvalue orbit’ by
$[x]_r := \left\{ y = \left( y^{(1)}, y^{(2)}, \ldots, y^{(N)} \right) : y^{(i)} \in [x^{(i)}]_{\mathcal{U}(i)} \text{ for all } i \right\}$.

We note that $[x]_r \subseteq [x]$ with equality when $\mathcal{V}$ is simple. To see an example, let $\mathcal{V} = \mathcal{R}^n = \mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}$ ($n > 1$). Recalling that $\{c_1, c_2, \ldots, c_n\}$ denotes the set of standard coordinate vectors in $\mathcal{R}^n$, we see that $[c_1] = \{c_1, c_2, \ldots, c_n\}$, while $[c_1]_r = \{c_1\}$.

Given a Jordan frame $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$, we assume that its listing/enumeration is fixed and define, for any $q = (q_1, q_2, \ldots, q_n) \in \mathcal{R}^n$,

$$q \ast \mathcal{E} := q_1 e_1 + q_2 e_2 + \cdots + q_n e_n.$$ 

We note that

$$\lambda(q \ast \mathcal{E}) = q^\dagger$$

and when $\mathcal{E}$ is fixed, $\Theta : q \mapsto q \ast \mathcal{E}$ is a continuous map from $\mathcal{R}^n$ to $\mathcal{V}$.

Recall that a nonempty set in $\mathcal{V}$ is a spectral set if it is of the form $\lambda^{-1}(Q)$ for some permutation invariant set $Q$ in $\mathcal{R}^n$. We will freely use the results in the following (easily verifiable) proposition.

**Proposition 2.1.** Let $P$ and $Q$ be nonempty subsets of $\mathcal{R}^n$ with $Q$ permutation invariant. Then,

- $Q = \Sigma_n(Q^\dagger)$.
- $x \in \lambda^{-1}(Q)$ if and only if $x = q \ast \mathcal{E}$ for some Jordan frame $\mathcal{E}$ and $q \in Q$.
- $\lambda(\lambda^{-1}(P)) = P \cap P^\dagger$ and $\lambda^{-1}(P) = \lambda^{-1}(P \cap P^\dagger)$. In particular, for any $\Omega \subseteq (\mathcal{R}^n)^\dagger$, $\lambda(\lambda^{-1}(\Omega)) = \Omega$ and $\lambda^{-1}(\Omega) = \lambda^{-1}(\Sigma_n(\Omega))$. Also, $\lambda^{-1}(Q) = \lambda^{-1}(Q^\dagger)$.
- Spectral sets can be generated by taking $\lambda$-inverse images of subsets of $(\mathcal{R}^n)^\dagger$. More generally, any set of the form $\lambda^{-1}(P) (= \lambda^{-1}(P \cap P^\dagger) = \lambda^{-1}(\Sigma_n(P \cap P^\dagger)))$ is a spectral set.
- The correspondence $\Omega \mapsto \lambda^{-1}(\Omega)$ is one-to-one and onto between nonempty subsets of $(\mathcal{R}^n)^\dagger$ and spectral sets in $\mathcal{V}$.
- For a Jordan frame $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$, the set $\Theta(Q) = \{q \ast \mathcal{E} : q \in Q\}$ is independent of the listing of elements in $\mathcal{E}$.
- $\lambda^{-1}(Q)$ is the union of sets of the form $\Theta(Q)$ as $\mathcal{E}$ varies over all Jordan frames.

Recall that an automorphism $\phi$ of $\mathcal{V}$ is an invertible linear transformation on $\mathcal{V}$ that satisfies the condition $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for all $x, y \in \mathcal{V}$. The set of all such transformations is denoted by $\text{Aut}(\mathcal{V})$ and we let $G$ denote the connected component of the identity transformation (henceforth called the component of identity) in $\text{Aut}(\mathcal{V})$. For example, $\text{Aut}(\mathcal{S}^n)$ consists of transformations of the form $\phi(X) := UXU^T$ ($X \in \mathcal{S}^n$) where $U \in \mathcal{R}^{n \times n}$ is an orthogonal matrix. When $U$ has determinant one, such a $\phi$ belongs to the corresponding $G$. Also, $\text{Aut}(\mathcal{H}^n)$ consists of transformations of the form $\phi(X) := UXU^*$ ($X \in \mathcal{H}^n$), where $U$ is a unitary matrix. In this case, $G = \text{Aut}(\mathcal{H}^n)$ ([10], page 15).
We need the following result connecting eigenvalue orbits and spectral sets.

**Proposition 2.2.**

(i) The following inclusions hold:

\[ \{ \phi(x) : \phi \in G \} \subseteq \{ \phi(x) : \phi \in \text{Aut}(\mathcal{V}) \} \subseteq [x] \quad (x \in \mathcal{V}). \]

These become equalities when \( \mathcal{V} \) is simple.

(ii) A set \( E \) in \( \mathcal{V} \) is a spectral set if and only if for all \( x \in E \), \([x] \subseteq E\).

(iii) If \( E \) is a spectral set in \( \mathcal{V} \), then for all \( \phi \in \text{Aut}(\mathcal{V}) \), \( \phi(E) = E \). Converse holds when \( \mathcal{V} \) is simple or \( \mathbb{R}^n \).

**Proof.** (i) As automorphisms preserve Jordan frames and eigenvalues, the stated inclusions follow. Now suppose \( \mathcal{V} \) is simple and let \( y \in [x] \) so that \( \lambda(x) = \lambda(y) \). We write the spectral decompositions \( x = \sum_1^n \lambda_i(x)e_i \) and \( y = \sum_1^n \lambda_i(y)f_i \), where \( \{e_1, e_2, \ldots, e_n\} \) and \( \{f_1, f_2, \ldots, f_n\} \) are Jordan frames in \( \mathcal{V} \). Since \( \mathcal{V} \) is simple, by Corollary IV.2.7 in [8], there is an automorphism in \( G \) that takes one Jordan frame to the other. Hence, we may write \( y = \phi(x) \) for some \( \phi \in G \). Thus, \([x] = \{\phi(x) : \phi \in G\}\).

(ii) This is given in Theorem 1, [16].

(iii) This is given in Theorem 2, [16]. \(\square\)

In view of Item (iii) in the above theorem, we see that in \( \mathcal{S}^n \), a set \( E \) is spectral if and only if

\( X \in E \Rightarrow UXU^T \in E \) for all orthogonal matrices \( U \in \mathcal{R}^{n \times n} \).

**3. Main connectedness results**

To motivate our discussion and results, we start with two examples.

**Example 1.** Let \( n > 1 \) and consider the non-simple algebra \( \mathcal{V} = \mathcal{R}^n \). Then, \( \text{Aut}(\mathcal{R}^n) = \Sigma_n \) and \( G \) consists of just the identity matrix. For any \( q \in \mathcal{R}^n \), \( \lambda(q) = q^\perp \). Also, for any permutation invariant set \( Q \) in \( \mathcal{R}^n \), \( \lambda^{-1}(Q) = Q \). Thus, in this setting, if a permutation invariant set is connected (arcwise connected), then its \( \lambda \)-inverse image is connected (respectively, arcwise connected). Can we improve this by assuming only the connectedness (arcwise connectedness) of \( Q^\perp \)? To answer this, let \( Q := \{c_1, c_2, \ldots, c_n\} \) denote the set of standard coordinate vectors in \( \mathcal{R}^n \). Then \( Q \) is permutation invariant and \( Q^\perp = \{c_1\} \). We see that while \( Q^\perp \) is connected (arcwise connected), \( \lambda^{-1}(Q) \) is not connected.

**Example 2.** Let \( n > 1 \) and consider the simple algebra \( \mathcal{V} = \mathcal{S}^n \). As noted previously, \( \text{Aut}(\mathcal{S}^n) \) consists of transformations of the form \( \phi(X) := UXU^T \) (\( X \in \mathcal{S}^n \)) where \( U \in \mathcal{R}^{n \times n} \) is an orthogonal matrix. Now consider the set \( Q := \{c_1, c_2, \ldots, c_n\} \) of Example 1.
Then, $Q$ is a permutation invariant set with $Q^1$ connected (arcwise connected). With $D$ denoting the diagonal matrix with $c_1$ on its diagonal, we have
\[
\lambda^{-1}(Q) = \lambda^{-1}(Q^1) = \left\{ UDUT : \ U \in \mathbb{R}^{n \times n} \text{ is orthogonal} \right\}
\]
\[
= \left\{ uu^T : \ u \in \mathbb{R}^n, \ ||u|| = 1 \right\}.
\]
(The vector $u$ that appears above is the first column of $U$.) As $n > 1$, the unit sphere in $\mathbb{R}^n$ is arcwise connected. Hence, $\lambda^{-1}(Q)$ is arcwise connected. Thus, in contrast to Example 1, a weaker hypothesis involving connectedness (arcwise connectedness) of $Q^1$ suffices to get the connectedness (arcwise connectedness) of $\lambda^{-1}(Q)$. We show below that in the setting of a simple algebra, such a statement holds for any permutation invariant set.

The following is a basic (possibly known) result about the connectedness of eigenvalue orbits. Recall that $G$ is the connected component of identity in $\text{Aut}(\mathcal{V})$.

**Theorem 3.1.** The following statements hold:

(a) $G$ is arcwise connected.
(b) When $\mathcal{V}$ is simple, for any $x \in \mathcal{V}$, $[x]$ is arcwise connected.
(c) For any $x \in \mathcal{V}$, $[x]_r$ is arcwise connected.

**Proof.** (a) As $\text{Aut}(\mathcal{V})$ is a (matrix) Lie group (see [8], page 36), its connected component of identity, namely $G$, is arcwise connected, see [6], Theorem 1.9.1.

(b) Now suppose $\mathcal{V}$ is simple and consider any $x \in \mathcal{V}$. Then, by Item (i) in the previous proposition, $[x] = \{ \phi(x) : \phi \in G \}$. As $G$ is arcwise connected and the map $\phi \mapsto \phi(x)$ is continuous, $[x]$ is also arcwise connected.

(c) If $\mathcal{V}$ is simple, $[x]_r = [x]$ for any $x$. Then, the result follows from (b). Suppose $\mathcal{V}$ is non-simple, and let $\mathcal{V}$ be a product of simple algebras: $\mathcal{V} = \mathcal{V}^{(1)} \times \mathcal{V}^{(2)} \times \cdots \times \mathcal{V}^{(N)}$, where each $\mathcal{V}^{(i)}$ is simple. Let $x = (x^{(1)}, x^{(2)}, \ldots, x^{(N)}) \in \mathcal{V}$. Then, by (b), each $[x^{(i)}]_{\mathcal{V}^{(i)}}$ is arcwise connected. Hence, the set
\[
[x]_r = [x^{(1)}]_{\mathcal{V}^{(1)}} \times [x^{(2)}]_{\mathcal{V}^{(2)}} \times \cdots \times [x^{(N)}]_{\mathcal{V}^{(N)}},
\]
being a product of arcwise connected sets, is arcwise connected. $\square$

To illustrate Items (b) and (c) in the above result, we let $x$ be a primitive idempotent in a simple algebra $\mathcal{V}$. Then, $[x]_r$, which is the set of all primitive idempotents, is arcwise connected. (This result is known, see [8], page 71.) This may fail if $\mathcal{V}$ is not simple: let $\mathcal{V} = \mathbb{R}^n$ ($n > 1$) and $x = c_1$. Then, $[x] = \{ c_1, c_2, \ldots, c_n \}$ is not connected. However, $[x]_r = \{ c_1 \}$ (a singleton set) is arcwise connected.
We now state a necessary condition for a $\lambda$-inverse image to be connected (arcwise connected).

**Proposition 3.2.** Let $P \subseteq \mathbb{R}^n$. If $\lambda^{-1}(P)$ is connected (arcwise connected) in $\mathcal{V}$, then $P \cap P^\perp$ is connected (arcwise connected) in $\mathbb{R}^n$. In particular, if $Q$ is permutation invariant and $\lambda^{-1}(Q)$ is connected (arcwise connected) in $\mathcal{V}$, then $Q^\perp$ is connected (arcwise connected) in $\mathbb{R}^n$.

**Proof.** The first statement follows from the continuity of the eigenvalue map $\lambda$ and the equality $P \cap P^\perp = \lambda(\lambda^{-1}(P))$. The second one follows from the equality $Q \cap Q^\perp = Q^\perp$. □

The next two results deal with sufficient conditions. They appear to be new even in the algebra $S^n$.

**Theorem 3.3.** Suppose $Q \subseteq \mathbb{R}^n$ is permutation invariant and one of the following conditions holds.

(a) $\mathcal{V}$ is simple and $Q^\perp$ is connected (arcwise connected).

(b) $Q$ is connected (arcwise connected).

Then $\lambda^{-1}(Q)$ is connected (respectively, arcwise connected) in $\mathcal{V}$.

**Proof.** Suppose condition (a) holds so that $\mathcal{V}$ is simple and $Q^\perp$ is connected (arcwise connected). We fix a Jordan frame $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ and consider the continuous map $\Theta : \mathbb{R}^n \to \mathcal{V}$ defined by $\Theta(q) := q \ast \mathcal{E}$, see Section 2. Then, the image $\Theta(Q^\perp)$, which is a subset of $\lambda^{-1}(Q^\perp)$, is connected (respectively, arcwise connected). Now let $x \in \lambda^{-1}(Q^\perp)$ be arbitrary. Then, $\lambda(x) \in Q^\perp$ and there is a Jordan frame $\{f_1, f_2, \ldots, f_n\}$ such that $x = \lambda_1(x)f_1 + \lambda_2(x)f_2 + \cdots + \lambda_n(x)f_n$. As $\mathcal{V}$ is simple, from Theorem 3.1, $[x]$ is arcwise connected. Also,

$$\lambda_1(x)e_1 + \lambda_2(x)e_2 + \cdots + \lambda_n(x)e_n \in \Theta(Q^\perp) \cap [x].$$

So the sets $\Theta(Q^\perp)$ and $[x]$ are connected (respectively, arcwise connected) and their intersection is nonempty. It follows that their union is also connected (respectively, arcwise connected). As $x$ is arbitrary in $\lambda^{-1}(Q^\perp)$, the connected component (respectively, arcwise connected component) of $\lambda^{-1}(Q^\perp)$ that contains $\Theta(Q^\perp)$ must be $\lambda^{-1}(Q^\perp)$ itself. This proves that $\lambda^{-1}(Q^\perp)$ is connected (respectively, arcwise connected) under the condition (a). The stated assertion follows since $\lambda^{-1}(Q) = \lambda^{-1}(Q^\perp)$.

Now suppose condition (b) holds. If $\mathcal{V}$ is simple, we can use the previous argument as $Q^\perp$ (which is the image of $Q$ under the continuous map $q \mapsto q^\perp$) is connected (arcwise connected). So, assume that $\mathcal{V}$ is non-simple and write it as an orthogonal direct sum $\mathcal{V} = \mathcal{V}^{(1)} \oplus \mathcal{V}^{(2)} \oplus \cdots \oplus \mathcal{V}^{(N)}$, where each $\mathcal{V}^{(i)}$ is simple. In $\mathcal{V}$, we fix a Jordan frame $\mathcal{E} := \{e_1, e_2, \ldots, e_n\}$ and consider the continuous map $\Theta : \mathbb{R}^n \to \mathcal{V}$ defined by $\Theta(q) = q \ast \mathcal{E}$. 
As $Q$ is connected (arcwise connected) and $\Theta$ is continuous, $\Theta(Q)$ is connected (arcwise connected). Also, as $Q$ is permutation invariant, any rearrangement/listing of elements of $\mathcal{E}$ will not alter the set $\Theta(Q)$. Hence, we may assume without loss of generality that $\mathcal{E} = \bigcup_{i=1}^{N} \mathcal{E}^{(i)}$, where each $\mathcal{E}^{(i)}$ is a Jordan frame in $\mathcal{V}^{(i)}$. Writing $q \in \mathcal{R}^{n}$ in the block form $q = (q^{(1)}, q^{(2)}, \ldots, q^{(N)})$, we see that $q \ast \mathcal{E} = q^{(1)} \ast \mathcal{E}^{(1)} + q^{(2)} \ast \mathcal{E}^{(2)} + \cdots + q^{(N)} \ast \mathcal{E}^{(N)}$. Now, let $x \in \lambda^{-1}(Q)$. Then, there exist a Jordan frame $\mathcal{F}$ in $\mathcal{V}$ and a $p \in Q$ such that $x = p \ast \mathcal{F}$. Similar to the above, we may write $x = p \ast \mathcal{F} = p^{(1)} \ast \mathcal{F}^{(1)} + p^{(2)} \ast \mathcal{F}^{(2)} + \cdots + p^{(N)} \ast \mathcal{F}^{(N)}$, where $\mathcal{F}^{(i)}$ is a Jordan frame in $\mathcal{V}^{(i)}$. Let $y := p \ast \mathcal{E} = p^{(1)} \ast \mathcal{E}^{(1)} + p^{(2)} \ast \mathcal{E}^{(2)} + \cdots + p^{(N)} \ast \mathcal{E}^{(N)}$. As $\lambda(p^{(i)} \ast \mathcal{E}^{(i)}) = \lambda(p^{(i)} \ast \mathcal{F}^{(i)})$ for all $i$, it follows that $y \in [x]_{\tau}$. Hence, $y \in \Theta(Q) \cap [x]_{\tau}$. As $\Theta(Q)$ is connected (arcwise connected) and $[x]_{\tau}$ is arcwise connected (by Theorem 3.1), it follows that $\Theta(Q) \cup [x]_{\tau}$ is connected (respectively, arcwise connected). Since $x$ is arbitrary in $\lambda^{-1}(Q)$, the connected (arcwise connected) component of $\lambda^{-1}(Q)$ must be $\lambda^{-1}(Q)$. This proves that $\lambda^{-1}(Q)$ is connected (respectively, arcwise connected). \[\square\]

**Corollary 3.4.** When $\mathcal{V}$ is simple, the following statements hold:

(i) If $\Omega$ is any subset of $(\mathcal{R}^{n})^\perp$ that is connected (arcwise connected), then $\lambda^{-1}(\Omega)$ is connected (respectively, arcwise connected).

(ii) If $P$ is any subset of $\mathcal{R}^{n}$ such that $P \cap P^\perp$ is connected (arcwise connected), then $\lambda^{-1}(P)$ is connected (respectively, arcwise connected).

**Proof.** (i) Let $\Omega$ be a connected (arcwise connected) subset of $(\mathcal{R}^{n})^\perp$. Then $Q := \Sigma_n(\Omega)$ is permutation invariant and $Q^\perp = \Omega$. Hence condition (a) in the above theorem applies. Since $\lambda^{-1}(\Omega) = \lambda^{-1}(Q)$, we see that $\lambda^{-1}(\Omega)$ is connected (arcwise connected).

(ii) Suppose $P$ is any subset of $\mathcal{R}^{n}$ such that $P \cap P^\perp$ is connected (arcwise connected). As $P \cap P^\perp$ is a subset of $(\mathcal{R}^{n})^\perp$ and $\lambda^{-1}(P) = \lambda^{-1}(P \cap P^\perp)$, the stated result follows from (i) applied to $\Omega := P \cap P^\perp$. \[\square\]

**Remarks.** (1) Suppose $\mathcal{V}$ is simple. In view of Proposition 3.2, for any set $P$ in $\mathcal{R}^{n}$, connectedness (arcwise connectedness) of $P \cap P^\perp$ is necessary and sufficient for that of $\lambda^{-1}(P)$. Thus, a spectral set $E$ in a simple algebra $\mathcal{V}$ is connected (arcwise connected) if and only if $\lambda(E)$ is connected (arcwise connected) in $\mathcal{R}^{n}$.

(2) The conclusions in the above corollary may fail if $\mathcal{V}$ is not simple. For example, in $\mathcal{R}^{3}$, $c_1 = (1, 0, 0)$ is on the boundary of $(\mathcal{R}^{3})^\perp$. Letting $\mathcal{V} = \mathcal{R}^{3}$ and $\Omega = \{c_1\}$, we see that $\lambda^{-1}(\Omega) = \{c_1, c_2, c_3\}$ (set of standard coordinate vectors in $\mathcal{R}^{3}$) is not connected. In the same setting, it is easy to construct an arcwise connected set $P$ such that $P \cap P^\perp = \{c_1\}$. (For example, $P$ could be a circle through $c_1$ such that $P \setminus \{c_1\}$ is outside of $(\mathcal{R}^{3})^\perp$.) For such a $P$, $\lambda^{-1}(P) = \lambda^{-1}(P \cap P^\perp) = \{c_1, c_2, c_3\}$ is not connected.

(3) Motivated by the above two results, one may ask if $\lambda$-inverse image of a simply connected permutation invariant set $Q$ in $\mathcal{R}^{n}$ is simply connected in $\mathcal{V}$. While the answer to this is not clear, we note that in $\mathcal{S}^{2}$, the set of all primitive idempotents is given by ([8], page 71).
\[
\left\{ \begin{array}{cc}
\cos^2 \theta & \cos \theta \sin \theta \\
\cos \theta \sin \theta & \sin^2 \theta 
\end{array} \right\}, \quad 0 \leq \theta \leq \pi.
\]

This is homeomorphic to a circle, hence not simply connected. However, it is of the form \( \lambda^{-1}(Q^\perp) \), where \( Q \) is the set of standard coordinate vectors in \( \mathcal{R}^2 \). While this \( Q \) is not simply connected, \( Q^\perp \), consisting of only one element, is simply connected. So the counterpart of (a) in Theorem 3.3 may not hold for simple connectedness.

We end this section by mentioning a classical result of Ky Fan. Suppose \( Q \) is a permutation invariant set that satisfies one of the conditions of Theorem 3.3. Assume further that \( Q \) is compact. Then, \( \lambda^{-1}(Q) \) is connected and (by Proposition 1.1) compact in \( \mathcal{V} \). Hence, for any \( c \in \mathcal{V} \), the image of \( \lambda^{-1}(Q) \) under the continuous function \( x \mapsto \langle c, x \rangle \) is connected and compact in \( \mathcal{R} \). So, there exist real numbers \( \delta \) and \( \Delta \) such that

\[
\left\{ \langle c, x \rangle : x \in \lambda^{-1}(Q) \right\} = [\delta, \Delta]. \tag{1}
\]

(With some additional work, it is possible to describe the forms of \( \delta \) and \( \Delta \).) To see a special case, consider two matrices \( C \) and \( A \) in \( \mathcal{H}^n \) and let \( Q = \Sigma_n(\{\lambda(A)\}) \). Clearly, \( Q \) is permutation invariant and compact (actually, finite). Since \( \mathcal{H}^n \) is simple and \( Q^\perp \) (a singleton set) is connected, by Theorem 3.3 (or by Theorem 3.1), \( \lambda^{-1}(Q) = [A] = \{U A U^* : U \in C^{n \times n} \text{ is unitary}\} \) is connected in \( \mathcal{H}^n \). As \( \langle X, Y \rangle = \text{tr}(XY) \) in \( \mathcal{H}^n \), (1) reads:

\[
\left\{ \text{tr}(CUA^*) : U \in C^{n \times n} \text{ is unitary}\right\} = [\delta, \Delta].
\]

With \( \Delta := \langle \lambda(C), \lambda(A) \rangle \) and \( \delta := \langle \lambda^\uparrow(C), \lambda(A) \rangle \), where \( \lambda^\uparrow(C) \) denotes the increasing rearrangement of \( \lambda(C) \), this statement is due to Fan [7] (see also [25], Corollary 1.6).

4. Components of a spectral set

We now describe connected (arcwise connected) components of a spectral set in a simple algebra.

**Theorem 4.1.** Let \( \mathcal{V} \) be simple and \( E := \lambda^{-1}(Q) \) be a spectral set in \( \mathcal{V} \), where \( Q \) is permutation invariant in \( \mathcal{R}^n \). Then, every connected (arcwise connected) component of \( E \) is a spectral set. Moreover, \( C \rightarrow \lambda(C) \) is a one-to-one correspondence between connected (arcwise connected) components of \( E \) and those of \( Q^\perp \).

**Proof.** We consider the case of connected components. The case of arcwise connected components is similar. Let \( C \) be a connected component of \( E \). As \( \mathcal{V} \) is simple, for any \( x \in C \), \([x]\) is (arcwise) connected (by Theorem 3.1) and \([x] \cap C \) contains \( x \). Hence, \([x] \subseteq C \). By Proposition 2.2(ii), \( C \) is a spectral set; we may now write \( C \) as \( C = \lambda^{-1}(P) \), where \( P \) is a permutation invariant set in \( \mathcal{R}^n \). As \( C \) is connected, its image \( \lambda(C) = P^\perp \).
is connected. We claim that this set is a connected component of $Q^\downarrow$. To simplify the notation, let $\Omega := P^\downarrow$. Then,

$C \subseteq E \Rightarrow P^\downarrow = \lambda(C) \subseteq \lambda(E) = Q^\downarrow \Rightarrow \Omega \subseteq Q^\downarrow.$

So, $\Omega$ is a connected subset of $Q^\downarrow$. Let $\Omega^*$ be the connected component of $Q^\downarrow$ that contains $\Omega$, so

$\Omega \subseteq \Omega^* \subseteq Q^\downarrow.$

Then,

$C = \lambda^{-1}(\Omega) \subseteq \lambda^{-1}(\Omega^*) \subseteq \lambda^{-1}(Q^\downarrow) = \lambda^{-1}(Q) = E.$

Since $\mathcal{V}$ is simple, by Item (i) in Corollary 3.4, $\lambda^{-1}(\Omega^*)$ is connected. By our assumption that $C$ is a connected component of $E$ we get $C = \lambda^{-1}(\Omega^*)$, leading to $\Omega = \lambda(C) = \lambda((\lambda^{-1}(\Omega^*))) = \Omega^*$, the last equality is due to Proposition 2.1. Hence, $\Omega$ is a connected component of $Q^\downarrow$.

Now we show that every connected component of $Q^\downarrow$ arises this way. Let $\Omega$ be a connected component of $Q^\downarrow$. By Item (i) in Corollary 3.4, $C := \lambda^{-1}(\Omega)$ is connected in $E$. Suppose $C^*$ is the connected component of $E$ that contains $C$. We show that $C = C^*$. Now,

$C \subseteq C^* \subseteq E \Rightarrow \lambda(C) \subseteq \lambda(C^*) \subseteq \lambda(E).$

We have $\lambda(C) = \lambda(\lambda^{-1}(\Omega)) = \Omega$, where the second equality is due to Proposition 2.1. It follows that $\Omega \subseteq \lambda(C^*) \subseteq Q^\downarrow$. As $\Omega$ is a connected component and $\lambda(C^*)$ is connected, we must have $\Omega = \lambda(C^*)$, that is, $\Omega = \lambda(C) = \lambda(C^*)$. So, $C = \lambda^{-1}(\Omega) = \lambda^{-1}(\lambda(C^*)) \supseteq C^*$. Since $C \subseteq C^*$, we conclude that $C = C^*$ and that $C$ is a connected component of $E$. Finally, the concluding statement in the theorem about one-to-one and onto correspondence is easily verified. \hfill \Box

Remarks. (4) The conclusions in the above result may fail if $\mathcal{V}$ is not simple, see Example 1.

5. Irreducibility of spectral cones

We now provide another application of Theorem 3.1. Recall that a (nonempty) set $K$ is a convex cone if it is convex and $tx \in K$ for all $x \in K$ and $t \geq 0$ in $\mathcal{R}$. If, in addition, $K \cap -K = \{0\}$, then $K$ is said to be a pointed convex cone. We say that a convex cone $K$ is reducible if it can be written as a sum $K = K_1 + K_2$ where $K_1$ and $K_2$ are nonzero convex cones with $\text{span}(K_1) \cap \text{span}(K_2) = \{0\}$. A convex cone that is not reducible is said to be irreducible. (These concepts hold in $\mathcal{R}^n$ as well.)
We now address the irreducibility of spectral cones. In $\mathcal{R}^n$, spectral cones (which are just permutation invariant convex cones) can be reducible or irreducible. In fact, $\mathcal{R}_+^n$ ($n > 1$) is reducible, while $\mathcal{R}_+^1$ is irreducible, where $1$ represents the vector of ones in $\mathcal{R}^n$. In a simple Euclidean Jordan algebra, the corresponding symmetric cone (= the cone of squares which is a closed convex self-dual homogeneous cone) is irreducible (see [8], Prop. III.4.5), even though it is of the form $\lambda^{-1}(\mathcal{R}_+^n)$ with $\mathcal{R}_+^n$ reducible for $n > 1$.

Below, we show that any pointed spectral cone in a simple Euclidean Jordan algebra is irreducible. First, we recall a well-known result ([11], slightly modified to suit our setting): If $K$ is a nonzero pointed reducible convex cone in $\mathcal{V}$, then there exists a unique set of nonzero irreducible convex cones $K_i$, $i = 1, 2, \ldots, r$, such that

$$K = K_1 + K_2 + \cdots + K_r$$

with $\text{span}(M) \cap \text{span}(N) = \{0\}$, whenever $M$ denotes the sum of some $K_i$s and $N$ denotes the sum of the rest of the $K_i$s. Moreover, the above representation is unique up to permutation of indices.

In this setting, we say that $K$ is a direct sum of $K_i$s.

**Theorem 5.1.** Suppose $\mathcal{V}$ is simple. Then, every pointed spectral cone in $\mathcal{V}$ is irreducible.

**Proof.** If possible, suppose $K$ is a nonzero pointed spectral cone in $\mathcal{V}$ which is a direct sum of irreducible convex cones:

$$K = K_1 + K_2 + \cdots + K_r$$

with $r > 1$. We claim that each $K_i$ is a pointed spectral cone. The pointedness of $K_i$ is clear, as $K$ is pointed. We show that $K_1$ is a spectral cone. Since $\mathcal{V}$ is simple, because of Items (i) and (ii) in Proposition 2.2, it is enough to show that for all $x \in K_1$, $[x] = \{\phi(x) : \phi \in G\} \subseteq K_1$. Let $\phi \in G$. Since $K$ is a spectral cone, $\phi(K) = K$ (by Item (iii) in Proposition 2.2). This implies that $K = \phi(K_1) + \phi(K_2) + \cdots + \phi(K_r)$ is another direct sum representation in terms of irreducible convex cones. By the uniqueness of factors in the decomposition, $\phi(K_1)$ must be equal to some $K_j$. So,

$$\phi(K_1) \subseteq \bigcup_{j=1}^{r} K_j, \forall \phi \in G.$$ 

Now, let $0 \neq x \in K_1$. Then, all the elements in $[x]$ are nonzero and

$$[x] = \{\phi(x) : \phi \in G\} \subseteq A \cup B,$$

where $A := K_1 \setminus \{0\}$ and $B := \bigcup_{j=2}^{r} K_j \setminus \{0\}$. Since $K_1 \cap B \subseteq \text{span}(K_1) \cap \text{span}(B) = \{0\}$, we see that $A \cap B = \emptyset$. (Here, ‘overline’ denotes the closure.) Similarly, $B \cap \overline{A} = \emptyset$. So the sets $A$ and $B$ are separated [21]. Now, by Theorem 3.3 (or by Theorem 3.1), $[x]$ is
connected. As \([x] \subseteq A \cup B\) and \(0 \neq x \in A\), we must have \([x] \subseteq A\). We conclude that \([x] \subseteq K_1\). This inclusion also holds for \(x = 0\) so

\[[x] \subseteq K_1, \forall x \in K_1.\]

Thus, \(K_1\) is a spectral cone by Item \((ii)\) in Proposition 2.2. A similar argument works for all other cones \(K_i\). Now, all the spectral cones \(K, K_1, K_2, \ldots, K_r\) are pointed spectral cones. Hence, by Theorem 7.3 in [15], \(e\) (the unit element of \(\mathcal{V}\)) or \(-e\) belongs to all of them. Since \(K\) is pointed, either \(e\) or \(-e\), but not both, can belong to all. Moreover, since the cones \(K_i\) have zero as the only common element, we conclude that \(r = 1\). This violates our assumption that \(r > 1\). Hence, \(K\) is irreducible. \(\Box\)

The above result gives an alternate proof of the fact that in any simple algebra, the symmetric cone is irreducible. Also, every pointed convex cone \(K\) in \(S^n\) satisfying the condition

\[X \in K \Rightarrow UXU^T \in K\]

for all orthogonal matrices \(U \in \mathcal{R}^{n \times n}\) is irreducible. We provide two more examples.

**Example 3.** Let \(n \geq 3\) and \(m \in \{1, 2, \ldots, n-1\}\). For each \(q \in \mathcal{R}^n\), let \(s_m(q)\) be the sum of the smallest \(m\) entries of \(q\). Then, the ‘rearrangement cone’

\[Q^n_m = \{q \in \mathcal{R}^n : s_m(q) \geq 0\},\]

is a permutation invariant pointed closed convex cone in \(\mathcal{R}^n\) [14]. Hence, by Item \((c)\) in Proposition 1.1,

\[\lambda^{-1}(Q^n_m) = \{x \in \mathcal{V} : \lambda_n(x) + \lambda_{n-1}(x) + \cdots + \lambda_{n-m+1}(x) \geq 0\}\]

is a pointed convex cone. (Note that when \(m = 1\), this cone is the symmetric cone of \(\mathcal{V}\).)

By the above theorem, this cone, in a simple algebra, is irreducible.

**Example 4.** Let \(n \geq 3\) and consider the following permutation invariant cone (see Example 2 in [15])

\[Q := \{q \in \mathcal{R}^n : tr(q) \geq \sqrt{\frac{n}{2}} \|q\|_2\}\]

where \(tr(q)\) denotes the sum of all entries of \(q\) and \(\|q\|_2\) denotes the 2-norm of \(q\). This cone is a proper cone (that is, it is a closed convex pointed cone with nonempty interior).

By the above theorem, when \(\mathcal{V}\) is a simple algebra of rank \(n\), the proper spectral cone

\[K := \lambda^{-1}(Q) = \{x \in \mathcal{V} : tr(x) \geq \sqrt{\frac{n}{2}} \|x\|_2\}\]
is irreducible, where $tr(x)$ denotes the sum of all eigenvalues of $x$ and $||x||_2 := ||\lambda(x)||_2$. Using the linearity of the trace and the strict convexity of $||\cdot||_2$, it is easy to show that every boundary vector in $K$ is an extreme vector. As $n$ (the rank of $\mathcal{V}$) is at least 3, such a property is false for the symmetric cone of $\mathcal{V}$ (take a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ and consider $e_1 + e_2$ which is on the boundary of the symmetric cone of $\mathcal{V}$, but not an extreme vector); so, $K$ and the symmetric cone of $\mathcal{V}$ are not isomorphic. This shows that in every simple algebra of rank $n \geq 3$, there is a proper (irreducible) spectral cone that is not isomorphic to the corresponding symmetric cone.

References