On the block norm-P property

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Abstract

A real $n \times n$ matrix $M$ is said to be a $P$-matrix if all its principal minors are positive. In a recent paper Chua and Yi [2] describe this property in terms of norm: There exists a $\gamma > 0$ such that for all nonnegative diagonal matrices $D$ and vectors $x$, $\|Mx + Dx\| \geq \gamma \|x\|$. In this paper, we introduce a block version of this property for a linear transformation defined on a product of normed or inner product spaces. In addition to relating this to (real) positive stability and positive principal minor properties, we study the invariance of this property by principal subtransformations and Schur complements. We also specialize this property to $Z$-transformations and to Euclidean Jordan algebras.

Key Words: Block norm-P property, Cartesian-P property, Uniform nonsingularity property (UNS), $Z$-transformation, Euclidean Jordan algebra, principal subtransformation, Schur complement.

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1 Introduction

A real $n \times n$ matrix $M$ is said to be a $P$-matrix if all its principal minors are positive. Ever since Fiedler and Ptak [5] introduced such matrices in 1962, they have been extensively studied and have become important in various fields, including matrix theory, optimization, dynamical systems, etc. Apart from the principal minor property, $P$-matrices have other equivalent properties, such as:

- The sign-reversal property, $x^* M x \leq 0 \Rightarrow x = 0$, where $'*'$ refers to the componentwise product and the inequality is with respect to the order induced by the nonnegative orthant.
- Every real eigenvalue of every principal submatrix of $M$ is positive.
- For every $q \in \mathbb{R}^n$, the linear complementarity problem $\text{LCP}(M, q)$ has a unique solution, where $\text{LCP}(M, q)$ denotes the problem of finding an $x \in \mathbb{R}^n$ such that $x \geq 0$, $y = Mx + q \geq 0$, and $\langle x, y \rangle = 0$.

If the matrix $M$ is a $Z$-matrix, that is, if all the off-diagonal entries of $M$ are non-positive, then the $P$-matrix property can be described in more than 50 equivalent ways, see [1].

Our motivation for the present work comes from a recent paper of Chua and Yi [2], where the following norm characterization is given as a special case of a much broader result:

A matrix $M \in \mathbb{R}^{n \times n}$ is a $P$-matrix if and only if there exists a $\gamma > 0$ such that for all nonnegative diagonal matrices $D \in \mathbb{R}^{n \times n}$, $\|Mx + Dx\| \geq \gamma \|x\| \quad \forall x \in \mathbb{R}^n$.

The proof given in [2] is of a variational nature and does not follow from simple ideas. One objective of the present paper is to give an alternative and possibly “simpler” proof and offer some connections to other results. In this paper, we consider a block version of the above result. To elaborate, consider a finite dimensional real normed space $V$ which is a direct product of spaces $V_i$, $i = 1, 2, \ldots, l$. Given a linear transformation $L$ on $V$, our objective is to study the following property which we call the block norm-$P$ property: There exists a $\gamma > 0$ such that $\|(L + \Lambda)x\| \geq \gamma \|x\|$ for all $x = (x_1, x_2, \ldots, x_l) \in V$, $\Lambda x = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_l x_l)$ with $x_i \in V_i$, $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, l$.

In this paper, we show that

- The Cartesian-$P$ property implies the block norm-$P$ property.
- The block norm-$P$ property is inherited by principal subtransformations and Schur complements.
- The block norm-$P$ property implies the positive principal minor property.
- The block norm-$P$ property implies the real positive stable property.

In addition, we describe the block norm-$P$ property for $Z$-transformations, and by studying the block norm-$P$ property in the setting of Euclidean Jordan algebras, we make connections to the uniform nonsingularity property [2].
2 Preliminaries

Throughout, $\mathbb{R}^n$ denotes the Euclidean $n$-space whose vectors are written as column vectors. The notations $\langle x, y \rangle$ and $x^Ty$ describe the usual inner product in $\mathbb{R}^n$. We use the symbol $\mathcal{S}^n$ to denote the space of all $n \times n$ real symmetric matrices and $\mathcal{S}^n_+$ for the set of all matrices in $\mathcal{S}^n$ that are also positive semidefinite.

In this paper, we let $V$ denote a real finite dimensional normed or inner product space, which in some cases can be an Euclidean Jordan algebra [4], [8] or a product of other spaces. For a set $S$ in $V$, we denote the interior and closure by $\text{int}(S)$ and $\overline{S}$, respectively. For a closed convex cone $K$ in $V$, we say that $K$ is proper if $K$ is pointed (that is, $K \cap -K = \{0\}$) and $\text{int}(K) \neq \emptyset$. If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, the dual of a closed convex cone $K$ is defined by

$$K^* = \{y \in V : \langle y, x \rangle \geq 0 \ \forall \ x \in K\}.$$

Now consider a linear transformation $L : V \to V$. We say that

1. $L$ is positive stable if every eigenvalue of $L$ has positive real part.
2. $L$ is real positive stable if every real eigenvalue of $L$ is positive.

When $(V, \langle \cdot, \cdot \rangle)$ is an inner product space with a proper cone $K$, we say that

(i) $L$ is a $\mathbb{Z}$-transformation on $K$ if

$$x \in K, \ y \in K^*, \ \text{and} \ \langle x, y \rangle = 0 \ \Rightarrow \ \langle L(x), y \rangle \leq 0.$$

(ii) $L$ is a Lyapunov-like transformation on $K$ if $L$ and $-L$ are both $\mathbb{Z}$-transformations on $K$.

(iii) $L$ is an $\mathbb{S}$-transformation on $K$ if there exists a $d \in \text{int}(K)$ such that $L(d) \in \text{int}(K)$.

We use the notation $L \in \mathbb{Z}(K)$ ($L \in \mathbb{S}(K)$) to mean that $L$ is a $\mathbb{Z}$-transformation (respectively, $\mathbb{S}$-transformation) on $K$. We recall the following result ([8], Theorem 7):

**Proposition 1** Let $V$ be a finite dimensional real inner product space and $K$ be a proper cone in $V$. If $L$ is a $\mathbb{Z}$-transformation on $K$, then the following are equivalent:

(i) $L$ is positive stable.

(ii) $L$ is real positive stable.

(iii) $L \in \mathbb{S}(K)$.

(iv) For every $q \in V$, the linear complementarity problem $\text{LCP}(L, K, q)$ has a solution, that is, there exists $x \in K$ such that $L(x) + q \in K^*$ and $\langle x, L(x) + q \rangle = 0$. 


Now suppose that $V = V_1 \times \cdots \times V_l$, where each $V_i$ is a normed (or an inner product) space. In this situation, we assume that the norm (or inner product) on $V$ is given by:

$$||x||^2 = \sum_{i=1}^{l} ||x_i||^2 \quad \text{or} \quad \langle x, y \rangle = \sum_{i=1}^{l} x_i y_i$$

for all $x = (x_1, x_2, \ldots, x_l)$ and $y = (y_1, y_2, \ldots, y_l)$ in $V$.

We say that a linear transformation $\Lambda : V \to V$ is a diagonal transformation if it is given by

$$\Lambda x = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_l x_l),$$

for some $(\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{R}^l$ and for all $x = (x_1, x_2, \ldots, x_l) \in V$. In this setting, we use the notation $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_l)$. We say that such a $\Lambda$ is a nonnegative diagonal transformation if all $\lambda_i$s are nonnegative.

Given $V = V_1 \times \cdots \times V_l$ and a linear transformation $L$ on $V$, recalling our earlier definition, we say that $L$ has the block norm-$P$ property if there exists a $\gamma > 0$ such that

$$||Lx|| \geq \gamma ||x||$$

(1)

for all $x \in V$ and for all nonnegative diagonal transformations $\Lambda$ on $V$.

For $V = V_1 \times \cdots \times V_l$ and linear $L$ on $V$, consider a nonempty set $\alpha \subseteq \{1, 2, \ldots, l\}$, written in the form $\alpha = \{i_1, i_2, \ldots, i_m\}$ with $i_1 < i_2 < \cdots < i_m$. Let $V_\alpha = V_1 \times V_{i_2} \times \cdots \times V_{i_m}$. We think of $V_\alpha$ as a subspace of $V$ in a canonical way and let $I_\alpha : V_\alpha \to V$ denote the inclusion map and $P_\alpha : V \to V_\alpha$ denote the projection map. Then $L_\alpha : V_\alpha \to V_\alpha$ defined by $L_\alpha = P_\alpha \circ L$ is a principal subtransformation of $L$ or a principal sub-block of $L$.

* If every $I_\alpha$ has positive determinant, we say that $L$ has the positive principal minor property.

Throughout the paper, while dealing with results on proper principal subtransformations, without loss of generality, we assume that $\alpha = \{1, 2, \cdots, m\}$ and write $V = V_1 \times V_2 \times \cdots \times V_l$ as $V = W_1 \times W_2$, where $W_1 := V_1 \times V_2 \times \cdots \times V_m$ and $W_2 = V_{m+1} \times V_{m+2} \times \cdots \times V_l$. With respect to the product $V = W_1 \times W_2$, we decompose $L$ in the form

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

(2)

where $A$ is a linear transformation from $W_1$ to itself, $B$ is a linear transformation from $W_2$ to $W_1$, etc. In this setting, $A$ is the principal subtransformation of $L$ corresponding to $\alpha$.

3 The Cartesian-$P$ property

It is well known that every $P$-matrix $M$ on $\mathbb{R}^n$ has the sign-reversal property

$$x_i(Mx)_i \leq 0 \quad \forall i = 1, 2, \ldots, n \Rightarrow x = 0,$$

or equivalently,

$$\max_{1 \leq i \leq n} x_i(Mx)_i > 0, \quad \forall 0 \neq x \in \mathbb{R}^n.$$
This property is extended to a linear transformation defined on a product of inner product spaces in [3]. Given inner product spaces $V_i$, $i = 1, 2, \ldots, l$, and $V = V_1 \times \cdots \times V_l$, a linear transformation $L$ on $V$ is said to have the Cartesian-$P$ property if there exists a $\gamma > 0$ such that for all $x = (x_1, x_2, \ldots, x_l)$ in $V$,

$$\max_{1 \leq i \leq l} \langle L_i(x), x_i \rangle \geq \gamma \| x \|^2,$$

where $L_i(x)$ is the $i$th component of $L(x)$. We note that when there is only one factor, the Cartesian-$P$ property reduces to the strong monotonocity of $L$:

$$\langle L(x), x \rangle \geq \gamma \| x \|^2 \quad \forall x \in V.$$

**Theorem 1** The Cartesian-$P$ property implies the block norm-$P$-property.

**Proof.** Assume that $L$ has the Cartesian-$P$ property on $V = V_1 \times \cdots \times V_l$ and let $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a nonnegative diagonal transformation. Fix $x = (x_1, x_2, \ldots, x_l)$ in $V$ with norm one and let, without loss of generality, $\langle L_1(x), x_1 \rangle = \max_{1 \leq i \leq l} \langle L_i(x), x_i \rangle \geq \gamma$. This implies that $\langle L_1(x) + \lambda_1 x_1, x_1 \rangle \geq \gamma$ for all $\lambda_1 \geq 0$. Now, as $1 \geq \| x_1 \|^2$, by the Cauchy-Schwarz inequality,

$$\| L_1(x) + \lambda_1 x_1 \| \geq \gamma \Rightarrow \| L_1(x) + \lambda_1 x_1 \|^2 \geq \| L_1(x) + \lambda_1 x_1 \| ^2 \| x_1 \|^2 \geq \gamma^2.$$

We further have

$$\| (L + \Lambda) x \|^2 = \sum_{i=1}^l \| L_i(x) + \lambda_i x_i \|^2 \geq \| L_1(x) + \lambda_1 x_1 \| ^2 \| x_1 \|^2 \geq \gamma^2.$$

Thus, $L$ has the block norm-$P$ property on $V$. $\square$

**4 The uniform nonsingularity property in Euclidean Jordan algebras**

Consider a Euclidean Jordan algebra $(V, \circ, \langle \cdot, \cdot \rangle)$ of rank $r$ (see [4] or [8]) and let $L : V \to V$ be a linear transformation. We say that $L$ has the uniform nonsingularity property (in short, the UNS-property) [2] if there exists a $\gamma > 0$ such that for all Jordan frames $F = \{e_1, e_2, \ldots, e_r\}$ in $V$, for any $x \in V$ with Peirce decomposition $x = \sum_{i \leq j} x_{ij}$ with respect to $F$, and all nonnegative numbers $\lambda_{ij}$, it holds that

$$\| L(x) + \sum_{i \leq j} \lambda_{ij} x_{ij} \| \geq \gamma \| x \|.$$

Suppose we fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ and consider the corresponding (orthogonal) Peirce decomposition of $V$ in the form

$$V = \sum_{i \leq j} V_{ij}.$$

Then with respect to this decomposition, the UNS-property implies the block norm-$P$ property. We record this in the following theorem.
Theorem 2 Suppose that $L$ has the UNS-property on a Euclidean Jordan algebra. Then with respect to any Peirce decomposition of $V$, $L$ has the block norm-$P$ property.

5 The real positive stable property

In this section, we consider the block norm-$P$ property when there is only one factor in $V$.

Theorem 3 Let $V$ be a finite dimensional real normed linear space and $L: V \rightarrow V$ be a linear transformation. Then following are equivalent:

(a) There exists $\gamma > 0$ such that
   \[||(L + \lambda I)x|| \geq \gamma ||x||, \forall x \in V \text{ and } \forall \lambda \geq 0.\] (3)

(b) $L$ is real positive stable.

(c) $L + \lambda I$ is invertible for all $\lambda \geq 0$.

(d) Every element in the interval $[I, L] := \{tL + (1 - t)I : t \in [0,1]\}$ is invertible.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are straightforward. Suppose $(c)$ holds. As $I$ is invertible and $tL + (1 - t)I = t[L + \frac{L - I}{t}I]$ is invertible for any $t \in (0,1]$, we have $(d)$. Now, suppose $(d)$ holds. To see that $(c)$ holds, take any $\lambda \geq 0$ and define $t = \frac{1}{1+\lambda}$. Then $L + \lambda I = \frac{1}{t}[tL + (1 - t)I]$ is invertible. Hence $(c)$ holds. We now show that $(c) \Rightarrow (a)$. Suppose $(c)$ holds and $(a)$ does not hold. Then for any natural number $k$, with $\gamma = \frac{1}{k}$, there exist $\lambda_k \geq 0$ and $||x_k|| = 1$ such that $||(L + \lambda_k I)x_k|| < \frac{1}{k}$. We assume without loss of generality, $x_k \rightarrow \overline{x}$, $||\overline{x}|| = 1$. We consider two cases:

Case 1: The sequence $\{\lambda_k\}$ is bounded. In this case, we let (without loss of generality) $\lambda_k \rightarrow \bar{\lambda}$ and get $||(L + \bar{\lambda} I)\overline{x}|| \leq 0$. This implies that $L + \bar{\lambda} I$ is singular, contradicting $(c)$.

Case 2: The sequence $\{\lambda_k\}$ is unbounded, say (without loss of generality), $\lambda_k \rightarrow \infty$. Then $\left|\frac{1}{\lambda_k}L + I\right|x_k \left| < \frac{1}{\lambda_k} \right|$ holds for all $k$, and so, $||\overline{x}|| = 0$. We reach a contradiction again. Hence we must have $(c) \Rightarrow (a)$.

We now state two immediate consequences. The first one follows from the fact that for a real linear transformation $L$, complex eigenvalues occur in conjugate pairs and the determinant of $L$ is the product of all its eigenvalues. The second one comes from taking $\Lambda$ to be a nonnegative multiple of the Identity transformation in the definition of the block norm-$P$ property.

Corollary 1 If $L$ satisfies one of the conditions of Theorem 3, then $\det(L) > 0$.

Corollary 2 Suppose $L$ has the block norm-$P$ property on $V = V_1 \times \cdots \times V_l$. Then $L$ is real positive stable. In particular, $\det(L) > 0$.

Our next corollary follows from combining the above theorem with Proposition 1.
Corollary 3 Let $V$ be a finite dimensional real inner product space and $K$ be a proper cone in $V$. If $L$ is a $Z$-transformation on $K$, then the conditions of Theorem 3 are further equivalent to each of the following:

(i) $L$ is positive stable.

(ii) $L \in S(K)$.

The following two examples illustrate the above corollary.

Example 1 For any $A \in R^{n \times n}$, consider the Lyapunov transformation $L_A$ defined on the space $S^n$ of all $n \times n$ real symmetric matrices by

$$L_A(X) := AX + XA^T.$$ 

It is known that both $L_A$ and $-L_A$ are $Z$-transformations on the semidefinite cone $S^n_+$ and $A$ is positive stable if and only if $L_A$ is positive stable ([7], Theorem 5). Now, by the above corollary, $A$ is positive stable if and only if

$$||AX + XA^T + \lambda X|| \geq \gamma ||X||, \forall X \in S^n \text{ and } \forall \lambda \geq 0,$$

where $||X||$ is the Frobenius norm of $X$.

Example 2 For any $A \in R^{n \times n}$, consider the Stein transformation $S_A$ defined on the space $S^n$ by

$$S_A(X) := X - AXA^T.$$ 

It is known that $S_A$ is a $Z$-transformation on the semidefinite cone $S^n_+$ and $A$ is Schur stable (that is, all eigenvalues of $A$ lie in the open unit disk) if and only if $S_A$ is positive stable ([6], Theorem 11). Thus, by the above corollary, $A$ is Schur stable if and only if

$$||X - AXA^T + \lambda X|| \geq \gamma ||X||, \forall X \in S^n \text{ and } \forall \lambda \geq 0.$$ 

6 An inheritance property

In this section, we prove the inheritance of the block norm-$P$ property by principal subtransformations and mention some consequences.

Theorem 4 Let $V = V_1 \times \cdots \times V_l$ be a normed linear space and $L$ be a linear transformation on $V$. Then $L$ has block norm-$P$ property if and only if every principal subtransformation of $L$ has the block norm-$P$ property. In particular, the block norm-$P$ property implies the positive principal minor property.
Proof. The “Only if” part: Assume that $L$ has the block norm-$\mathbf{P}$ property. We need to show that every (proper) principal subtransformation of $L$ has the block norm-$\mathbf{P}$ property. To this end, let, without loss of generality, $\alpha = \{1, 2, \ldots, m\}$ with $1 \leq m < l$ and write $L$ in the block form (2). We show that $A$ has the block norm-$\mathbf{P}$ property on $V_1 \times \cdots \times V_m$. Consider an arbitrary nonnegative diagonal transformation $\Gamma = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ on $V_1 \times \cdots \times V_m$ and $u \in V_1 \times \cdots \times V_m$. Corresponding to any $\mu > 0$, let $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m, \mu, \mu, \ldots, \mu)$, $v = -Cu$ and $x := [u, v]_\mu^T$. Then the inequality
\[
\|((L + \Lambda)x\| \geq \gamma \|x\|
\]
implies
\[
\left\| \begin{bmatrix}
(A + \Gamma)u + B(v) \\
D(v) 
\end{bmatrix} \right\| \geq \gamma \left\| \begin{bmatrix}
u \\
\frac{v}{\mu} \end{bmatrix} \right\|.
\]

Letting $\mu \to \infty$, we get $\|(A + \Gamma)u\| \geq \gamma \|u\|$. This shows that $A$ has the block norm-$\mathbf{P}$ property. In particular, by Corollary 1, $\det A > 0$. Thus we have proved that every principal subtransformation of $L$ has the block norm-$\mathbf{P}$ property and that $L$ has the positive principal minor property. Finally, the “If” part is obvious. \hfill \Box

We now show that for a square matrix on $\mathbb{R}^n$ (considered as a product of $n$ copies of $\mathbb{R}$), the block norm-$\mathbf{P}$ property is the same as the $\mathbf{P}$-property, thereby giving an alternate proof of a result of Chua and Yi [2].

Theorem 5 Suppose $M \in \mathbb{R}^{n \times n}$. Then the following are equivalent.

(a) There exists $\gamma > 0$ such that for all nonnegative diagonal matrices $\Lambda$ on $\mathbb{R}^n$
\[
\|(M + \Lambda)x\| \geq \gamma \|x\|, \forall x \in \mathbb{R}^n.
\]

(b) Every principal submatrix $M_\alpha$ of $M$ is real positive stable.

(c) $M$ has the sign-reversal property: $x^* M x \leq 0 \Rightarrow x = 0$.

(d) $\forall \alpha \subseteq \{1, 2, \ldots, n\}, M_\alpha + \varepsilon I_\alpha$ is invertible for all $\varepsilon \geq 0$.

Proof. The implication (a) $\Rightarrow$ (b) follows from Theorem 4 and Corollary 2. The equivalence of (b) and (c) is well known, see the Introduction.

(c) $\Rightarrow$ (a): Suppose that (c) holds and (a) fails. Then there exist sequences $x_k$ in $\mathbb{R}^n$ and $\Lambda_k$ (nonnegative diagonal matrices) such that $\|x_k\| = 1$, $\|Mx_k + \Lambda_k x_k\| < \frac{1}{k}$ for all $k = 1, 2, \ldots$. Let $y_k := Mx_k + \Lambda_k x_k$. Then $x_k^* M x_k = x_k^* y_k - x_k^* \Lambda_k x_k \leq x_k^* y_k$. Let, without loss of generality, $x_k \to x$. Then $\|x\| = 1$. Since $y_k \to 0$, we have $x^* M x \leq 0$. From condition (c) we get $x = 0$, which is a contradiction.

The equivalence (b) $\Leftrightarrow$ (d) follows from Theorem 3. \hfill \Box

The result below is a consequence of Theorems 4 and 5.
Corollary 4 Consider a normed linear space \( V = V_1 \times V_2 \times \cdots \times V_r \times V_{r+1} \times \cdots \times V_l \) with \( V_1 = V_2 = \cdots = V_r = R \). Let \( L \) be a linear transformation on \( V \). If \( L \) has the block norm-\( P \) property, then the \( r \times r \) principal subtransformation, viewed as a matrix, is a \( P \)-matrix.

As an application of this corollary, consider an Euclidean Jordan algebra of rank \( r \) with a given Jordan frame \( \{e_1, e_2, \ldots, e_r\} \). Then the corresponding Peirce decomposition of \( V \) is given by

\[
V = Re_1 + Re_2 + \cdots + Re_r + \sum_{i<j} V_{ij}.
\]

Since this is an orthogonal decomposition, we can form an orthogonal basis of \( V \). Then a linear transformation on \( V \), when viewed as a matrix with respect to this basis, will have its leading principal \( r \times r \) block given by the matrix

\[
M = \begin{bmatrix}
\langle L(e_i), e_j \rangle \\
\|e_j\|^2
\end{bmatrix}.
\]

The above corollary says that if \( L \) has the block norm-\( P \) property on \( V = Re_1 + Re_2 + \cdots + Re_r + \sum_{i<j} V_{ij} \), the matrix \( M \) is a \( P \)-matrix. In particular, we have the following which generalizes an earlier result in [10] proved in the setting of \( S^n \):

Corollary 5 Suppose \( V \) is a Euclidean Jordan algebra of rank \( r \) and \( L \) has the UNS-property on \( V \). Then, for any Jordan frame \( \{e_1, e_2, \ldots, e_r\} \), the matrix \( M \) (given above) is a \( P \)-matrix.

Theorem 6 For a linear transformation \( L \) on \( V = V_1 \times V_2 \times \cdots \times V_l \), the following are equivalent:

(a) \( L \) has the block norm-\( P \) property on \( V \).

(b) For every nonnegative diagonal transformation \( \Lambda \), every principal subtransformation of \( L + \Lambda \) is invertible.

Proof. (a) \( \Rightarrow \) (b): Suppose (a) holds. Then for any nonnegative diagonal transformation \( \Lambda \), \( L + \Lambda \) also has the block norm-\( P \) property on \( V \). From Theorem 4, we see that any principal subtransformation of \( L + \Lambda \) also has the block norm-\( P \) property and hence (via Corollary 2) invertible.

(b) \( \Rightarrow \) (a): Assume (b) and suppose that (a) is false. Then there exist sequences \( \Lambda_k = \text{diag}(\lambda_1^{(k)}, \ldots, \lambda_i^{(k)}) \) and \( x_k \) with \( \|x_k\| = 1 \) such that \( \|(L + \Lambda_k)x_k\| < \frac{1}{k} \) for all \( k \). We consider the following cases:

Case 1: For every \( i \), the sequence \( \lambda_i^{(k)} \) is bounded. Without loss of generality, we assume that \( \lambda_i^{(k)} \to \lambda_i \) and \( x_k \to x \). Then we have \( \|(L + \Lambda)x\| \leq 0 \), where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_i) \). This implies that \( x = 0 \), which contradicts \( \|x\| = 1 \).

Case 2: For every \( i \), \( \lambda_i^{(k)} \) is unbounded. Without loss of generality, let \( \lambda_i^{(k)} \to \infty \) for all \( i \) and \( x_k = (x_1^{(k)}, \ldots, x_l^{(k)}) \to \bar{x} \). Let \( L_1(x_k) \) be the first component of \( L(x_k) \). Then \( \frac{1}{\lambda_i^{(k)}} L_1(x_k) + x_1^{(k)} \leq 1 \). Hence, we have \( x_1^{(k)} \to \bar{x} = 0 \). Similarly, we have \( \bar{x} = 0 \). Thus, \( \bar{x} = 0 \). This contradicts \( \|\bar{x}\| = 1 \).
Case 3: Without loss of generality, we assume that $\lambda_i^{(k)} \to \lambda_i$ for $i = 1, \ldots, m$ and $\lambda_i^{(k)} \to \infty$ for $i = m + 1, \ldots, l$. We write $L$ in the block form $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and let $x_k = \begin{bmatrix} x_{k1} \\ x_{k2} \end{bmatrix}$, where $x_{k1} = (x_1^{(k)}, \ldots, x_m^{(k)}) \to x_1$ and $x_{k2} = (x_{m+1}^{(k)}, \ldots, x_l^{(k)}) \to x_2$. Then

$$\left\| \begin{bmatrix} Ax_{k1} + Bx_{k2} + \Lambda_1^{(k)} x_{k1} \\ Cx_{k1} + Dx_{k2} + \Lambda_2^{(k)} x_{k2} \end{bmatrix} \right\| < \frac{1}{k},$$

where $\Lambda_1^{(k)} := \text{diag}(\lambda_1^{(k)}, \ldots, \lambda_m^{(k)}) \to \Lambda_1 := \text{diag}(\lambda_1, \ldots, \lambda_m)$ and $\Lambda_2^{(k)} := \text{diag}(\lambda_{m+1}^{(k)}, \ldots, \lambda_l^{(k)})$. Using the same argument as in Case 2, we have $x_{k2} \to x_2 = 0$. Then we have $\|Ax_1 + \Lambda_1 x_1\| = 0$. This implies, from (b), that $x_1 = 0$. Thus, $x = 0$. This contradicts $\|x\| = 1$. □

The following example shows that the invertibility of $L + \Lambda$ alone is not enough to get the block norm-$P$ property.

Example 3 Let $V = R \times R$ and

$$L = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$ 

Then it is easy to verify that $L + \Lambda$ is invertible for any nonnegative diagonal matrix $\Lambda$ but $L$ is not a P-matrix. By Theorem 4, $L$ does not have the block norm-$P$ property.

7 The inheritance of block norm-$P$ property by Schur complements

Consider a linear transformation $L$ on $V = V_1 \times V_2 \times \cdots \times V_l$. Let $W_1 := V_1 \times V_2 \times \cdots \times V_m$ and $W_2 = V_{m+1} \times V_{m+2} \times \cdots \times V_l$ for $1 \leq m < l$. With respect to the product $V = W_1 \times W_2$, we decompose $L$ in the form (2).

Theorem 7 Let $V$ and $L$ be as given above. Consider the following statements:

(a) $L$ has the block norm-$P$ property on $V = V_1 \times V_2 \times \cdots \times V_l$.

(b) $A$ has the block norm-$P$ property on $W_1 = V_1 \times V_2 \times \cdots \times V_m$, $D$ has the block norm-$P$ property on $W_2 = V_{m+1} \times V_{m+2} \times \cdots \times V_l$, and for all $\varepsilon \geq 0$, $D - C(A + \varepsilon I)^{-1} B$ has the block norm-$P$ property on $W_2 = V_{m+1} \times V_{m+2} \times \cdots \times V_l$.

(c) $A$, $D$, $D - C(A + \varepsilon I)^{-1} B$ are real positive stable for all $\varepsilon \geq 0$.

Then

$$(a) \Rightarrow (b) \Rightarrow (c).$$

The implication $(c) \Rightarrow (a)$ holds when $l = 2$ and $m = 1$. 10
Proof. (a) ⇒ (b): Assume that \( L \) has the block \( P \)-property on \( V \). Then, by Theorem 4, \( A \) and \( D \) have the block norm-\( P \) property. We fix \( \varepsilon \geq 0 \) and let
\[
N := A + \varepsilon I.
\]
To see that \( D - CN^{-1}B \) has the block norm-\( P \) property on \( W_2 \), we proceed as follows. For any \( x \in W_1 \times W_2 \), we write \( x = \begin{bmatrix} u \\ v \end{bmatrix} \). Let \( \Lambda_1 \) be a nonnegative diagonal transformation on \( W_1 \) and \( \Lambda_2 \) be a nonnegative diagonal transformation on \( W_2 \). Then from
\[
\left\| \begin{pmatrix} L + \begin{bmatrix} \Lambda_1 + \varepsilon I & 0 \\ 0 & \Lambda_2 \end{bmatrix} \\ N \end{bmatrix} x \right\| \geq \gamma \|x\|,
\]
we have
\[
\left\| \begin{pmatrix} Nu + Bv + \Lambda_1 u \\ Cu + Dv + \Lambda_2 v \end{pmatrix} \right\| \geq \gamma \left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|.
\]
Let \( Nu + Bv = p \). Then \( u = N^{-1}p - N^{-1}Bv \). Thus,
\[
\left\| \begin{pmatrix} p + \Lambda_1 (N^{-1}p - N^{-1}Bv) \\ CN^{-1}p + (D - CN^{-1}B)v + \Lambda_2 v \end{pmatrix} \right\| \geq \gamma \left\| \begin{pmatrix} N^{-1}p - N^{-1}Bv \\ v \end{pmatrix} \right\|.
\]
Fix \( v \) and take \( u = -N^{-1}Bv \) so that \( p = 0 \). Hence,
\[
\left\| \begin{pmatrix} -\Lambda_1 N^{-1}Bv \\ (D - CN^{-1}B)v + \Lambda_2 v \end{pmatrix} \right\| \geq \gamma \left\| \begin{pmatrix} -N^{-1}Bv \\ v \end{pmatrix} \right\|.
\]
Taking \( \Lambda_1 = 0 \), we get \( \| (D - CN^{-1}B)v + \Lambda_2 v \| \geq \gamma \|v\| \). Thus, \( D - CN^{-1}B \) has the block \( P \)-property on \( W_2 \).

(b) ⇒ (c): This implication holds since the block norm-\( P \) property implies real positive stability property, see Corollary 2.

(c) ⇒ (a): We prove this when \( l = 2 \) and \( m = 1 \). In this case, \( V = V_1 \times V_2 \); we continue to write
\[
L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]
Assume (c). Our goal is to show that
\[
\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{bmatrix} \lambda I_1 & 0 \\ 0 & \mu I_2 \end{bmatrix} \right\| \geq \gamma \|x\|, \forall x \in V \text{ and } \forall \lambda, \mu \geq 0.
\]
Suppose that this result does not hold. Then there exist nonnegative sequences \( \{\lambda_k\}, \{\mu_k\}, \) and \( \{x_k\} \), where \( \|x_k\| = 1 \) and \( x_k = \begin{bmatrix} u_k \\ v_k \end{bmatrix} \) for all \( k \) such that
\[
\left\| \begin{pmatrix} (A + \lambda_k I)u_k + Bv_k \\ Cu_k + (D + \mu_k I)v_k \end{pmatrix} \right\| < \frac{1}{k}.
\]
Without loss of generality, let \( x_k \to \vec{x} = \begin{bmatrix} \vec{u} \\ \vec{v} \end{bmatrix} \) with \( \|\vec{x}\| = 1 \).

We consider the following cases:

**Case 1:** \( \{\lambda_k\} \) and \( \{\mu_k\} \) are bounded. Without loss of generality, let \( \lambda_k \to \bar{\lambda} \) and \( \mu_k \to \bar{\mu} \). Then

\[
\left\| \begin{bmatrix} (A + \bar{\lambda}I)\vec{u} + B\vec{v} \\ C\vec{u} + (D + \bar{\mu}I)\vec{v} \end{bmatrix} \right\| \leq 0.
\]

As \( (A + \bar{\lambda}I)\vec{u} + B\vec{v} = 0 \) and \( A \) is real positive stable, we have \( \vec{u} = -(A + \bar{\lambda}I)^{-1}B\vec{v} \), from which we get \( (D - C(A + \bar{\lambda}I)^{-1}B)\vec{v} + \bar{\mu}\vec{v} = 0 \). This implies that \( -\bar{\mu} \) is a nonpositive eigenvalue of \( D - C(A + \bar{\lambda}I)^{-1}B \). This is a contradiction.

**Case 2:** \( \{\lambda_k\} \) and \( \{\mu_k\} \) are unbounded. Without loss of generality, let \( \lambda_k \to \infty \) and \( \mu_k \to \infty \).

Then we have

\[
\left\| \begin{bmatrix} \lambda_k \left( \frac{1}{\lambda_k} A + I \right) u_k + \frac{1}{\lambda_k} Bv_k \\ \mu_k \left( \frac{1}{\mu_k} C u_k + \left( \frac{1}{\mu_k} D + I \right) v_k \right) \end{bmatrix} \right\| < \frac{1}{k}.
\]

From this we get,

\[
\left\| \left( \frac{1}{\lambda_k} A + I \right) u_k + \frac{1}{\lambda_k} Bv_k \right\| < \frac{1}{k\lambda_k}
\]

and

\[
\left\| \frac{1}{\mu_k} C u_k + \left( \frac{1}{\mu_k} D + I \right) v_k \right\| < \frac{1}{k\mu_k}.
\]

Taking limits, we see that \( \vec{u} = 0 \) and \( \vec{v} = 0 \). This is a contradiction as \( \|\vec{x}\| = 1 \).

**Case 3:** \( \{\lambda_k\} \) is bounded and \( \{\mu_k\} \) is unbounded. Without loss of generality, let \( \lambda_k \to \bar{\lambda} \) and \( \mu_k \to \infty \). We have

\[
\left\| \begin{bmatrix} (A + \lambda_k I)u_k + Bv_k \\ \mu_k \left( \frac{1}{\mu_k} C u_k + \left( \frac{1}{\mu_k} D + I \right) v_k \right) \end{bmatrix} \right\| < \frac{1}{k}.
\]

Thus,

\[
\| (A + \lambda_k I)u_k + Bv_k \| < \frac{1}{k}
\]

and

\[
\left\| \frac{1}{\mu_k} C u_k + \left( \frac{1}{\mu_k} D + I \right) v_k \right\| < \frac{1}{k\mu_k}.
\]

Taking limits, we get \( \vec{v} = 0 \) and \( (A + \bar{\lambda}I)\vec{u} = 0 \) with \( \vec{u} \neq 0 \). This implies that \( -\bar{\lambda} \) is a nonpositive eigenvalue of \( A \). This is a contradiction.

**Case 4:** \( \{\lambda_k\} \) is unbounded and \( \{\mu_k\} \) is bounded. Without loss of generality, let \( \lambda_k \to \infty \) and \( \mu_k \to \bar{\mu} \).

As in Case 3, we get \( (D + \bar{\mu}I)\vec{v} = 0 \) with \( \vec{v} \neq 0 \). This implies that \( -\bar{\mu} \) is a nonpositive eigenvalue of \( D \). This is a contradiction. Thus, we must have \( (c) \Rightarrow (a) \). This completes the proof of the theorem.
8 The block norm-P property for Z-transformations

For our final result below, consider, for each \( i = 1, 2, \ldots, l \), a proper cone \( K_i \) in the finite dimensional real inner product space \( V_i \). We let \( V = V_1 \times V_2 \times \cdots \times V_l \) and \( K = K_1 \times K_2 \times \cdots \times K_l \).

**Theorem 8** Suppose that \( L \in Z(K) \) on \( V = V_1 \times V_2 \times \cdots \times V_l \), where \( K = K_1 \times K_2 \times \cdots \times K_l \).

Then the following are equivalent:

(a) \( L \) has the block norm-P property on \( V \).

(b) \( L + \Lambda \) is invertible for all nonnegative diagonal transformations \( \Lambda \) on \( V \).

**Proof.** (a) \( \Rightarrow \) (b): When (a) holds, \( L + \Lambda \) has the block norm-P property for every nonnegative diagonal transformation \( \Lambda \). Corollary 2 now shows that \( L + \Lambda \) is invertible.

(b) \( \Rightarrow \) (a): Assume that (b) holds and let \( \Lambda \) be a nonnegative diagonal transformation on \( V \).

We show that any principal subtransformation of \( L + \Lambda \) is invertible and apply Theorem 6 to get (a). Without loss of generality, assume that the chosen principal subtransformation corresponds to the set \( \alpha = \{1, 2, \ldots, m\} \) and write \( V_\alpha := V_1 \times V_2 \times \cdots \times V_m \) and \( K_\alpha := K_1 \times K_2 \times \cdots \times K_m \).

From (b), \( L + \lambda I \) is invertible for all \( \lambda \geq 0 \); thus, \( L \in Z(K) \cap S(K) \) by Corollary 3. As \( L \in S(K) \), there is a \( d \in int(K) \) with \( L(d) \in int(K) \). Then \( (L + \lambda I)d \in int(K) \) (as \( \Lambda \) is a nonnegative diagonal transformation and each component \( d_i \) of \( d \) is in \( int(K_i) \)). Consequently, \( L + \Lambda \in Z(K) \cap S(K) \).

Now, by Theorem 2 in [11],

\[
L + \Lambda \in Z(K) \cap S(K) \Rightarrow (L + \Lambda)_\alpha \in Z(K_\alpha) \cap S(K_\alpha).
\]

From Proposition 1, we see that \( (L + \Lambda)_\alpha \) is invertible. Now we can quote Theorem 6 and complete the proof. \( \square \)

The following example shows that the \( Z \)-property is crucial in the above theorem.

**Example 4** Consider the matrix \( M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). It is easily verified that for all \( \varepsilon_1, \varepsilon_2 \geq 0 \), the matrix

\[
\begin{bmatrix}
0 + \varepsilon_1 & 1 \\
-1 & 0 + \varepsilon_2
\end{bmatrix}
\]

is invertible, yet \( M \) does not have the block norm-P property on \( V = R \times R \).

**Concluding Remarks.** In this paper, we introduced the concept of block norm-P property for a linear transformation on a product of finite dimensional real normed or inner product spaces. We showed that this property is inherited by principal subtransformations and by Schur complements. We also specialized this property to \( Z \)-transformations and to Euclidean Jordan algebras. We remark that Proposition 1 and Theorem 8 may be relevant in the study of complementarity problems and Theorem 7 may be relevant in linear dynamical systems (see Section 6.3, [9]).
References


