Math. Program., Ser. B
DOI 10.1007/s10107-007-0159-8

FULL LENGTH PAPER

Z-transformations on proper and symmetric cones

Z-transformations

M. Seetharama Gowda · Jiyuan Tao

Received: 6 February 2006 / Accepted: 15 December 2006

Abstract Motivated by the similarities between the properties of **Z**-matrices on R^n_+ and Lyapunov and Stein transformations on the semidefinite cone $\mathcal{S}^n_+$, we introduce and study **Z**-transformations on proper cones. We show that many properties of **Z**-matrices extend to **Z**-transformations. We describe the diagonal stability of such a transformation on a symmetric cone by means of quadratic representations. Finally, we study the equivalence of **Q** and **P** properties of **Z**-transformations on symmetric cones. In particular, we prove such an equivalence on the Lorentz cone.

Keywords Z-transformation · Symmetric cone · Quadratic representation · Diagonal stability

Mathematics Subject Classification (2000) 90C33 · 17C55 · 15A48 · 37B25

This paper is dedicated to Professor Steve Robinson on the occasion of his 65th birthday. His ideas and results greatly influenced a generation of researchers including the first author of this paper.
Steve, thanks for being a great teacher, mentor, and a friend. Best wishes for a long and healthy productive life.

M. S. Gowda (✉)
Department of Mathematics and Statistics, University of Maryland,
Baltimore County, Baltimore, MD 21250, USA
e-mail: gowda@math.umbc.edu

J. Tao
Department of Mathematical Sciences, Loyola College in Maryland,
Baltimore, MD 21210, USA
e-mail: jtao@loyola.edu

1 Introduction

A real $n \times n$ matrix M is said to be a **Z**-matrix if all its off-diagonal entries are non-positive [12]. Such matrices appear in various fields including differential equations, dynamical systems, optimization, economics, etc., see [3]. For our discussion, we recall the following equivalent properties for a **Z**-matrix ([3], Chap. 6):

(1) M is a **P**-matrix, i.e., all the principal minors of M are positive.
(2) M is a **Q**-matrix, i.e., for any $q \in R^n$, the *linear complementarity problem* LCP(M, q) has a solution; this means that there exists an $x \in R^n$ such that

$$x \geq 0, \; y = Mx + q \geq 0, \text{ and } \langle x, y \rangle = 0.$$

(3) There exists a $d > 0$ such that $Md > 0$.
(4) M is invertible and M^{-1} is (entrywise) nonnegative.
(5) M is positive stable, that is, every eigenvalue of M has positive real part.
(6) M is diagonally stable, i.e., there exists a positive diagonal matrix D such that $MD + DM^T$ is positive definite.

(This is a partial list. See [3] for 50 equivalent conditions, and [21] for more.)

In the above properties, $x \geq 0$ means that $x \in R^n_+$ (the nonnegative orthant) and $d > 0$ means that d is in the interior of R^n_+. We remark that the **P**-matrix property in (1) can also be equivalently described by

$$x * (Mx) \leq 0 \Rightarrow x = 0$$

where $x * (Mx)$ denotes the componentwise product of vectors x and Mx [6]. Also, when M is a **P**-matrix, the solution x in (2) is unique for all $q \in R^n$.

In recent times, various authors have studied complementarity problems over symmetric cones, especially over the positive semidefinite cone and the second-order cone. In [15] and [14], the following result was established. Let $\mathcal{S}^n$ denote the space of all real $n \times n$ symmetric matrices and $\mathcal{S}^n_+$ denote the cone of positive semidefinite matrices in $\mathcal{S}^n$. For a real $n \times n$ matrix A, let the Lyapunov and Stein transformations be respectively given by

$$L_A(X) = A \circ X := AX + XA^T \quad \text{and} \quad S_A(X) = X - AXA^T \quad (X \in \mathcal{S}^n).$$

Let L denote either L_A or S_A. Then the following are equivalent:

(a) L has the **P**-property on $\mathcal{S}^n$, i.e.,

$$\left. \begin{array}{r} X \quad \text{and} \quad L(X) \quad \text{(operator) commute} \\ X \circ L(X) \preceq 0 \end{array} \right\} \Rightarrow X = 0.$$

(b) L has the **Q**-property with respect to the cone $\mathcal{S}^n_+$, i.e., for any $Q \in \mathcal{S}^n$, there exists an $X \in \mathcal{S}^n$ such that

$$X \succeq 0, \; Y = L(X) + Q \succeq 0, \; \text{and} \; \langle X, Y \rangle = 0.$$

(c) There exists a matrix $D \succ 0$ in $\mathcal{S}^n$ such that $L(D) \succ 0$.
(d) L is invertible and $L^{-1}(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$.
(e) L is positive stable, i.e., the real part of any eigenvalue of L is positive.

Here $X \succeq 0$ means that $X \in \mathcal{S}^n_+$ and $D \succ 0$ means that D lies in the interior of $\mathcal{S}^n_+$. Motivated by the similarities between the above two results, one may ask if the Lyapunov and Stein transformations have some sort of a **Z**-property on the semidefinite cone, or more generally, whether one could define and study **Z**-transformations on, say, symmetric cones and proper cones. The main objective of this paper is to undertake such a study and extend properties (1–6) to proper/symmetric cones.

Consider a real finite dimensional Hilbert space V and let K be a proper cone in V, that is, K is a closed convex cone in V with $K \cap (-K) = \{0\}$ and $K - K = V$. Let the dual of K be defined by

$$K^* := \{x \in V : \langle x, y\rangle \geq 0 \text{ for all } y \in K\}.$$

Following [31], we say that a linear transformation $T : V \rightarrow V$ has the *cross-positive property with respect to* K if

$$x \in K,\ y \in K^*,\ \text{ and } \langle x, y\rangle = 0 \Rightarrow \langle T(x), y\rangle \geq 0;$$

Now we say that a linear transformation $L : V \rightarrow V$ has the **Z***-property with respect to K if $-L$ has the cross-positive property.* Cross-positive transformations have been studied in [2,3,8,9,19,31,35,37]. These works deal with equivalent formulations and eigenvalue properties of cross-positive transformations. Other pertinent references include [32,33], and [34].

In this paper, we show that Lyapunov and Stein transformations (and numerous others) have the **Z**-property on the semidefinite cone, and that many of the properties of **Z**-matrices generalize to proper cones. In particular, we focus on the complementarity properties.

We mention here two other generalizations of the **Z**-matrix property. In [4], Borwein and Dempster extend the **Z**-matrix property to vector lattices, primarily to study order linear complementarity problems and describe least element solutions of such problems. Given a vector lattice E and a linear transformation $T : E \rightarrow E$, they define the type (Z) property by the condition

$$x \wedge y = 0 \Rightarrow (Tx) \wedge y \leq 0$$

where $x \wedge y$ denotes the vector minimum of x and y in E. In [7], Cryer and Dempster define, on a real Hilbert space H which is also a vector lattice, by

$$x \wedge y = 0 \Rightarrow \langle T(x), y\rangle \leq 0.$$

In the case of a Hilbert lattice, see Sect. 5 in [4], this definition coincides with the above definition of **Z**-transformation. As our cones, to the most part, are non-latticial, we will not consider these generalizations in this paper.

Here is an outline of the paper. In Sect. 2, we will cover the basic material dealing with the complementarity properties, degree theory, and Euclidean Jordan algebras. In Sect. 3, we introduce **Z**-transformations on proper cones, give examples, and extend properties (2–5) of the Introduction; in particular, we show that for a **Z**-transformation, the **Q** and **S** properties with respect to K (which are the cone analogs of conditions (2) and (3) of the Introduction) are equivalent. Results in this section show that a linear transformation $L : V \rightarrow V$ has the **Z** and **Q** properties with respect to K if and only if the dynamical system

$$\frac{dx}{dt} + L(x) = 0$$

is globally asymptotically stable (which means that its trajectory, starting at any point in V, converges to zero) and viable with respect to K (which means the trajectory, starting at any point in K, stays in K). Partly motived by the simultaneous stability problem in dynamical systems, in Sect. 4 we deal with the question of when the sum and product of **Z**-transformations have the **Q**-property. In Sect. 5, we consider Euclidean Jordan algebras, and prove an analog of Item (6) of the Introduction replacing the positive diagonal matrix by a quadratic representation. Section 6 deals with the **P**-property on a Euclidean Jordan algebra which is defined by

$$\left.\begin{array}{r} x \circ L(x) \leq 0 \\ x \quad \text{operator commutes with} \quad L(x) \end{array}\right\} \Rightarrow x = 0,$$

and the question whether **Q** and **P** properties are equivalent for a **Z**-transformation; in particular, we prove this equivalence on the Euclidean Jordan algebra $\mathcal{L}^n$ whose corresponding symmetric cone is the Lorentz cone.

2 Preliminaries

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional real Hilbert space. For any set C in V, we write C° for the interior of C; when C is closed convex, we write $\Pi_C(x)$ for the (orthogonal) projection of x onto C.

Throughout this paper, we assume that K denotes a closed convex cone in V, i.e., K is closed, $K + K \subseteq K$ and $\lambda K \subseteq K$ for all $\lambda \geq 0$. Given K, we define the dual cone by

$$K^* := \{y \in V : \langle y, x \rangle \geq 0 \text{ for all } x \in K\}.$$

K is said to be *proper* if

$$K \cap (-K) = \{0\} \quad \text{and} \quad K - K = V,$$

or equivalently, interiors of K^* and K are nonempty ([3], p. 2).

Examples of proper cones include self-dual cones (those satisfying $K^* = K$) and (in particular) symmetric cones, see Sect. 2.3 below.

We note that

$$0 \neq x \in K^* \quad \text{and} \quad d \in K^\circ \Rightarrow \langle x, d \rangle > 0.$$

A closed convex cone F contained in K is said to be a *face* of K if the following is satisfied:

$$\left.\begin{array}{r} x \in K,\ \ y \in K,\ \text{ and } \ \lambda x + (1-\lambda) y \in F \\ for \quad \text{some} \quad \lambda \in (0,1) \end{array}\right\} \Rightarrow x \in F \text{ and } y \in F.$$

Given a linear transformation $L : V \to V$ and a face F of K, the transformation $L_{FF} : Span(F) \to Span(F)$ defined by

$$L_{FF}(x) = \Pi_{Span(F)}(L(x)) \quad (x \in Span(F))$$

is called a *principal subtransformation of* L (induced by F). We say that L has the *positive principal minor property* with respect to K if for every face F of K, the determinant of L_{FF} is positive. We remark that when $V = R^n$, $K = R^n_+$, and L is an $n \times n$ real matrix, the positive principal minor property coincides with the **P**-matrix property.

We recall that for any two matrices A and B in $R^{n\times n}$, $tr(AB) = \sum_{i,j} a_{ij} b_{ij} = tr(BA)$.

2.1 LCP Concepts

Recall that K is a closed convex cone in V. Given a linear transformation $L : V \to V$ and $q \in V$, *the cone linear complementarity problem*, LCP(L, K, q), is to find an $x \in V$ such that

$$x \in K,\ y = L(x) + q \in K^*,\ \text{ and } \langle x, y \rangle = 0.$$

This problem is a particular case of a variational inequality problem [10].

The following concepts are defined with respect to the cone K.

(i) We say that L has the **Q**-*property with respect to* K (and write $L \in \mathbf{Q}(K)$) if LCP(L, K, q) has a solution for all $q \in V$,

(ii) L has the $\mathbf{R}_0$-*property with respect to* K if zero is the only solution of LCP$(L, K, 0)$, and

(iii) L is said to have the **S**-*property with respect to* K (and write $L \in \mathbf{S}(K)$) if there exists a d such that

$$d \in K^\circ \quad \text{and} \quad L(d) \in K^\circ.$$

We note that when L is invertible, L has the **Q**-property with respect to K if and only if L^{-1} has the **Q**-property with respect to K^*. (This is because, x is a solution of LCP$(L^{-1}, K^*, -L^{-1}(q))$ if and only if $y := L^{-1}(x) - L^{-1}(q)$ is a solution of LCP(L, K, q).)

2.2 Degree theory

Given $L : V \to V$ and $q \in V$, we define the mapping $F_q : V \to V$ by

$$F_q(x) := x - \Pi_K(x - L(x) - q).$$

It is well known that $F_q(\overline{x}) = 0$ if and only if $\overline{x}$ solves LCP(L, K, q) [10].

We use degree theory concepts/results to prove existence of solutions of equations. We refer to [24] and [10] for further details. The following are standard results in complementarity theory, see Theorem 2.5.10 and its proof in [10].

Theorem 1 *Suppose L has the $\mathbf{R}_0$-property with respect to K and let $F_0(x) := x - \Pi_K(x - L(x))$. Then for any bounded open set Ω containing zero in V, deg$(F_0, \Omega, 0)$ is defined. If this degree is nonzero, then L has the $\mathbf{Q}$-property with respect to K.*

Theorem 2 (Karamardian [22]) *Suppose that $(K^*)^o$ is nonempty and there is a vector $e \in (K^*)^o$ such that zero is the only solution of the problems LCP$(L, K, 0)$ and LCP(L, K, e). Then L has the $\mathbf{Q}$-property with respect to K.*

We mention two consequences of Karamardian's Theorem. First, if K is proper, L is copositive on K (which means that $\langle L(x), x\rangle \geq 0$ for all $x \in K$) and L has the $\mathbf{R}_0$-property on K, then L has the $\mathbf{Q}$-property on K. Second, if L is positive definite on V (which means $\langle L(x), x\rangle > 0$ for all $x \neq 0$), then L has the $\mathbf{Q}$-property with respect to K.

2.3 Euclidean Jordan algebras

In this subsection, we recall some concepts, properties, and results from Euclidean Jordan algebras. Most of these can be found in Faraut and Korányi [11] and Gowda, Sznajder and Tao [18].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle \cdot, \cdot \rangle)$ where $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space over R and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in V$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in V$ where $x^2 := x \circ x$, and
(iii) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$ for all $x, y, z \in V$.

We also assume that there is an element $e \in V$ (called the *unit* element) such that $x \circ e = x$ for all $x \in V$.

In V, the set of squares

$$K := \{x \circ x : x \in V\}$$

is a *symmetric cone* (see Faraut and Korányi [11], p. 46). This means that K is a self-dual closed convex cone and for any two elements $x, y \in K^\circ$, there exists an invertible linear transformation $\Gamma : V \to V$ such that $\Gamma(K) = K$ and $\Gamma(x) = y$.

An element $c \in V$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, e_2, \ldots, e_m\}$ of primitive idempotents in V is a *Jordan frame* if

$$e_i \circ e_j = 0 \text{ if } i \neq j, \text{ and } \sum_1^m e_i = e.$$

Note that $\langle e_i, e_j \rangle = \langle e_i \circ e_j, e \rangle = 0$ whenever $i \neq j$.

Theorem 3 (The Spectral decomposition theorem) (Faraut and Korányi [11]) *Let V be a Euclidean Jordan algebra. Then there is a number r (called the rank of V) such that for every $x \in V$, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ with*

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{1}$$

If x in (1) is invertible (which means that every λ_i is nonzero), we define

$$x^{-1} := \lambda_1^{-1} e_1 + \cdots + \lambda_r^{-1} e_r$$

and when $x \geq 0$ (or equivalently, every λ_i is nonnegative), we define

$$x^{\frac{1}{2}} = \lambda_1^{\frac{1}{2}} e_1 + \cdots + \lambda_r^{\frac{1}{2}} e_r.$$

Remark 1 R^n is a Euclidean Jordan algebra with inner product and Jordan product defined respectively by

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i \quad \text{and} \quad x \circ y = x * y.$$

Here R^n_+ is the corresponding symmetric cone.

Remark 2 $\mathcal{S}^n$, the set of all $n \times n$ real symmetric matrices, is a Euclidean Jordan algebra with the inner and Jordan product given by

$$\langle X, Y \rangle := \text{trace}(XY) \quad \text{and} \quad X \circ Y := \frac{1}{2}(XY + YX).$$

In this setting, the symmetric cone K is the set of all positive semidefinite matrices in $\mathcal{S}^n$ denoted by $\mathcal{S}^n_+$.

Remark 3 Consider R^n $(n > 1)$ where any element x is written as

$$x = \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix}$$

with $x_0 \in R$ and $\overline{x} \in R^{n-1}$. The inner product in R^n is the usual inner product. The Jordan product $x \circ y$ in R^n is defined by

$$x \circ y = \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \overline{y} \end{bmatrix} := \begin{bmatrix} \langle x, y \rangle \\ x_0 \overline{y} + y_0 \overline{x} \end{bmatrix}.$$

We shall denote this Euclidean Jordan algebra $(R^n, \circ, \langle \cdot, \cdot \rangle)$ by $\mathcal{L}^n$. In this algebra, the cone of squares, denoted by $\mathcal{L}^n_+$, is called the *Lorentz cone* (or the second-order cone). It is given by

$$\mathcal{L}^n_+ = \{x : ||\overline{x}|| \leq x_0\}.$$

We note the spectral decomposition of any x with $\overline{x} \neq 0$:

$$x = \lambda_1 e_1 + \lambda_2 e_2$$

where

$$\lambda_1 := x_0 + ||\overline{x}||, \;\; \lambda_2 := x_0 - ||\overline{x}||$$

and

$$e_1 := \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\overline{x}}{||\overline{x}||} \end{bmatrix}, \text{ and } e_2 := \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\overline{x}}{||\overline{x}||} \end{bmatrix}.$$

In a Euclidean Jordan algebra V, for a given $x \in V$, we define the corresponding *Lyapunov transformation* $L_x : V \to V$ by

$$L_x(z) = x \circ z.$$

We say that elements x and y *operator commute* if $L_x L_y = L_y L_x$. It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame (Faraut and Korányi [11], Lemma X.2.2).

We recall the following from Gowda, Sznajder and Tao [18] (with the notation $x \geq 0$ when $x \in K$):

Proposition 1 *For* $x, \; y \in V$, *the following conditions are equivalent*:

1. $x \geq 0$, $y \geq 0$, *and* $\langle x, y \rangle = 0$.
2. $x \geq 0$, $y \geq 0$, *and* $x \circ y = 0$.

In each case, elements x *and* y *operator commute.*

The Peirce decomposition Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in a Euclidean Jordan algebra V. For $i, j \in \{1, 2, \ldots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V : x \circ e_i = x\} = R\, e_i$$

and when $i \neq j$,

$$V_{ij} := \left\{ x \in V : x \circ e_i = \frac{1}{2} x = x \circ e_j \right\}.$$

Then we have the following

Theorem 4 (Theorem IV.2.1, Faraut and Korányi [11]) *The space V is the orthogonal direct sum of spaces V_{ij} $(i \leq j)$. Furthermore,*

$$\begin{array}{l} V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj} \\ V_{ij} \circ V_{jk} \subset V_{ik} \text{ if } i \neq k \\ V_{ij} \circ V_{kl} = \{0\} \text{ if } \{i, j\} \cap \{k, l\} = \emptyset. \end{array}$$

Thus, given any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we have the Peirce decomposition of $x \in V$:

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$$

where $x_i \in R$ and $x_{ij} \in V_{ij}$.

A Euclidean Jordan algebra is said to be *simple* if it is not a direct sum of two Euclidean Jordan algebras. The classification theorem (Chap. V, Faraut and Korányi [11]) says that every simple Euclidean Jordan algebra is isomorphic to one of the following:

(1) The algebra $\mathcal{S}^n$ of $n \times n$ real symmetric matrices.
(2) The algebra $\mathcal{L}^n$.
(3) The algebra $\mathcal{H}_n$ of all $n \times n$ complex Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
(4) The algebra $\mathcal{Q}_n$ of all $n \times n$ quaternion Hermitian matrices with (real) trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
(5) The algebra $\mathcal{O}_3$ of all 3×3 octonion Hermitian matrices with (real) trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

The following result characterizes all Euclidean Jordan algebras.

Theorem 5 (Faraut and Korányi [11], Proposition III.4.4, III.4.5, Theorem V.3.7) *Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras.*

Suppose V is a Euclidean Jordan algebra with rank r. Given any matrix $A \in R^{r \times r}$ and a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V, we define a transformation $R_A : V \to V$ as follows. For any $x \in V$, write the Peirce decomposition

$$x = \sum_{1}^{r} x_i e_i + \sum_{i<j} x_{ij}.$$

Then

$$R_A(x) := \sum_1^r y_i e_i + \sum_{i<j} x_{ij},$$

where

$$[y_1, y_2, \ldots, y_r]^T = A\left([x_1, x_2, \ldots, x_r]^T\right).$$

Intuitively, we can think of R_A as the map that (linearly) transforms the element x by changing its 'diagonal'. It turns out that properties of A and R_A are closely related, see Example 5 below.

3 Definition of Z-property, examples, and general results

Definition 1 Let K be a proper cone in V. Given a linear transformation $L : V \to V$, we say that L has the **Z**-property with respect to K (and write $L \in \mathbf{Z}(K)$) if

$$x \in K,\ y \in K^*, \ \text{ and } \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0.$$

When the context is clear, we simply write $\mathbf{Z}$ in place of $\mathbf{Z}(K)$.

We immediately note that $\mathbf{Z}(K)$ is closed under finite nonnegative linear combinations; also, if $L \in \mathbf{Z}(K)$, then $L^T \in \mathbf{Z}(K^*)$.

Example 1 When $V = R^n$ and $K = R^n_+$, it is easily verified that an $n \times n$ real matrix has the **Z**-property with respect to R^n_+ if and only if it is a **Z**-matrix (that is, all its off-diagonal entries are nonpositive).

Example 2 Let $V = \mathcal{S}^n$, $K = \mathcal{S}^n_+$, and write $X \succeq 0$ when $X \in \mathcal{S}^n_+$. For $A \in R^{n\times n}$, recall that the Lyapunov transformation L_A is given by

$$L_A(X) = AX + XA^T.$$

We claim that L_A has the **Z**-property with respect to $\mathcal{S}^n_+$. Suppose $X \succeq 0$, $Y \succeq 0$, and $\langle X, Y\rangle = 0$. Then, $XY = 0 = YX$ [15] and hence

$$\langle L_A(X), Y\rangle = \langle AX + XA^T, Y\rangle = tr(AXY) \\ + tr(XA^TY) = 0 + tr(YXA^T) = 0.$$

Example 3 Let K be any proper cone in a general V. If $S : V \to V$ is linear and $S(K) \subseteq K$, then it is trivial to see that $-S \in \mathbf{Z}(K)$. Moreover,

$$t \in R,\ S(K) \subseteq K \Rightarrow L = tI - S \in \mathbf{Z}(K).$$

In particular, if $V = \mathcal{S}^n$, $K = \mathcal{S}^n_+$, and $A \in R^{n\times n}$, the Stein transformation S_A, defined by

$$S_A(X) = X - AXA^T$$

has the **Z**-property on $\mathcal{S}^n_+$. In [26], Mesbahi and Papavassilopoulos study transformations of the form

$$L(X) = X - \sum_1^k B_i X B_i^T.$$

Calling such a transformation a type-**Z**-transformation, they study the rank minimization problem with applications to control theory. We note that this transformation is a particular case of

$$L(X) = AX + XA^T - \sum_1^k B_i X B_i^T$$

which is in $\mathbf{Z}(\mathcal{S}^n_+)$.

Example 4 Let $V = \mathcal{L}^n$ and $J := \text{diag}(1, -1, -1, \ldots, -1) \in R^{n\times n}$. We claim that $A \in \mathbf{Z}(\mathcal{L}^n_+)$ *if and only if there exists* $\gamma \in R$ *such that* $\gamma J - (JA + A^T J)$ *is positive semidefinite on* $\mathcal{L}^n$.

Since the 'if' part is easily verified, we prove the 'only if' part. Assume $A \in \mathbf{Z}(\mathcal{L}^n_+)$ and let $\langle x, Jx\rangle = 0$. Then $u := \pm x$ is on the boundary of $\mathcal{L}^n_+$, $v := Ju \in \mathcal{L}^n_+$ and $\langle u, v\rangle = 0$. By the **Z**-property of A, we must have $\langle Au, v\rangle \le 0$. Expressing this last inequality in terms of x, we get the implication

$$\langle x, Jx\rangle = 0 \Rightarrow \langle -(JA + A^T J)x, x\rangle \ge 0.$$

By a result of Finsler [13] (see also [39]), there exists a $\gamma \in R$ such that $\gamma J - (JA + A^T J)$ is positive semidefinite on $\mathcal{L}^n$. This proves the claim.

We note that the above claim (expressed in terms of exponential nonnegativity) appears in [35] with a different proof.

Example 5 Let V be a Euclidean Jordan algebra with rank r and K be the corresponding cone of squares. Let $A = (a_{ij}) \in R^{r\times r}$. Consider R_A defined with respect to a Jordan frame $\{e_1, \ldots, e_r\}$ (see the end of Sect. 2). Then we have the following:

- (i) If R_A has the **Z**-property with respect to K, then A is a **Z** matrix.
- (ii) If A is a **Z**-matrix with $a_{ii} \le 1$ for all i, then R_A has the **Z**-property with respect to K.

Item (i) can be seen by noting, for any Jordan frame $\{e_1, \ldots, e_r\}$, $a_{ij}||e_i||^2 = \langle R_A(e_j), e_i\rangle \le 0$ for $i \ne j$. To see (ii), assume that A is a **Z**-matrix with $a_{ii} \le 1$ for all i. Let $x, y \in K$ and $\langle x, y\rangle = 0$. We write the Peirce decompositions $x =$

$\sum_1^r x_i e_i + \sum_{i<j} x_{ij}$ and $y = \sum_1^r y_i e_i + \sum_{i<j} y_{ij}$. Then by the properties of spaces V_{ij}, $x_i \geq 0$, $y_i \geq 0$ (for all $i = 1, \ldots, r$) and

$$\left\langle \sum_{i<j} x_{ij}, \sum_{i<j} y_{ij} \right\rangle = -\sum_1^r x_i y_i ||e_i||^2 \leq 0.$$

Now $R_A(x) = \sum_1^r \mu_i e_i + \sum_{i<j} x_{ij}$, where

$$[\mu_1, \mu_2, \ldots, \mu_r]^T = A\left([x_1, x_2, \ldots, x_r]^T\right).$$

Then

$$\langle R_A(x), y\rangle = \sum_1^r y_i \mu_i ||e_i||^2 - \sum_1^r x_i y_i ||e_i||^2 = \sum_1^r (a_{ii} - 1) x_i y_i ||e_i||^2 + \sum_{i \neq j} a_{ij} x_j y_j ||e_j||^2 \leq 0.$$

Thus R_A has the **Z**-property with respect to K.

The following example shows that R_A need not have the **Z**-property when $a_{ii} > 1$. Let $V = \mathcal{S}^2$, $K = \mathcal{S}^2_+$,

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Define R_A with respect to the Jordan frame $\{E_1, E_2\}$. Letting

$$X = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

and

$$Y = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix},$$

we see that $X \succeq 0$, $Y \succeq 0$, and $\langle X, Y\rangle = 0$. However,

$$R_A(X) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{2} \end{bmatrix}$$

and $\langle R_A X, Y\rangle = \frac{3}{8} > 0$. Hence R_A does not have the **Z**-property with respect to $\mathcal{S}^2_+$ even though A is a **Z**-matrix. □

As mentioned in the Introduction, a **Z**-transformation is the negative of a cross-positive transformation. Hence we have the following equivalent properties on any proper cone [9,31]:

(a) L has the **Z**-property with respect to K.
(b) $e^{-tL}(K) \subseteq K$ for all $t \geq 0$.
(c) For any $x_0 \in K$, the trajectory of the dynamical system

$$\frac{dx}{dt} + L(x) = 0; \;\; x(0) = x_0$$

stays in K.
(d) $L = \lim(\alpha_n I - S_n)$ where $\alpha_n \in R$ and S_n is a linear transformation on V with $S_n(K) \subseteq K$ for all n.

In the above list, Item (d) describes how one can generate **Z**-transformations. Transformations of the form

$$\alpha I - S \quad \text{with} \quad S(K) \subseteq K$$

are especially important in matrix theory, see [2] and [3]. Any such transformation will be called an **M**-transformation if (spectral radius) $\rho(S) \leq \alpha$, and a nonsingular **M**-transformation if $\rho(S) < \alpha$; for properties of such transformations, see Chap. 2, Sect. 4 in [2].

Remark 4 When $V = R^n$ with $K = R^n_+$, every **Z**-matrix is of the form $\alpha I - S$ where S is a nonnegative matrix. In the general case, this need not be true: The following example shows that **Z**-transformations need not be of the form $\alpha I - S$ with $S(K) \subseteq K$.

Example 6 Consider L_A on $\mathcal{S}^2$ where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

We know that L_A has the **Z**-property with respect to $\mathcal{S}^n_+$, see Example 2. Suppose that there exists $\alpha \in R$ and $S(\mathcal{S}^2_+) \subseteq \mathcal{S}^2_+$, such that $L_A = \alpha I - S$. Then $S = \alpha I - L_A$. For any $X, Y \in \mathcal{S}^2_+$ with $\langle X, Y\rangle > 0$, we have

$$\langle S(X), Y\rangle \geq 0 \Rightarrow \langle \alpha X, Y\rangle - \langle L_A(X), Y\rangle \geq 0 \Rightarrow \alpha \geq \frac{\langle L_A(X), Y\rangle}{\langle X, Y\rangle}.$$

Now take

$$X_k = \begin{bmatrix} 1 & \frac{1}{\sqrt{k}} \\ \frac{1}{\sqrt{k}} & \frac{1}{k} \end{bmatrix} \quad \text{and} \quad Y_k = \begin{bmatrix} \frac{1}{k^2} & 0 \\ 0 & \frac{1}{k} \end{bmatrix}.$$

It is easy to see that

$$tr(X_k Y_k) = \frac{2}{k^2} \to 0, \ \ tr(L_A(X_k)Y_k) = \frac{4}{k^2} + \frac{2}{k\sqrt{k}} \to 0,$$

while

$$\frac{\langle L_A(X_k), Y_k\rangle}{\langle X_k, Y_k\rangle} = 2 + \sqrt{k} \to \infty$$

as $k \to \infty$, which is clearly a contradiction.

Our next result extends properties (2–4) of the Introduction to proper cones. In the classical (matrix) situation, equivalence of properties (2–4) is invariably proved via property (1), see e.g., Theorem 3.11.10 in [6], Theorem 1.5.2 in [1], or Page 134, [3]. This cannot be done in the general case as the so-called positive principal minor property fails to hold, see Remark 8.

Theorem 6 *Let K be a proper cone in V and suppose $L : V \to V$ has the* **Z***-property with respect to K. Then the following conditions are equivalent*:

(a) *L has the* **Q***-property with respect to K.*
(b) *L^{-1} exists and $L^{-1}(K) \subseteq K$ (equivalently, $L^{-1}(K^\circ) \subseteq K^\circ$).*
(c) *There exists $d \in K^\circ$ such that $L(d) \in K^\circ$.*
(d) *L^T has the* **Q***-property with respect to K^*.*
(e) *$(L^T)^{-1}$ exists and $(L^T)^{-1}(K^*) \subseteq K^*$ (equivalently, $(L^T)^{-1}((K^*)^o) \subseteq (K^*)^o$).*
(f) *There exists $u \in (K^*)^o$ such that $L^T(u) \in (K^*)^o$.*

Note: In view of a Theorem of Alternative ([3], p. 9), Item (f) is equivalent to:

$$-L(x) \in K, x \in K \Rightarrow x = 0.$$

Proof (a) $\Rightarrow$ (b): Assume (a). Take any $q \in K^\circ$, and consider a solution x of LCP$(L, K, -q)$ so that

$$x \in K, \ \ y = L(x) - q \in K^*, \quad \text{and} \quad \langle x, y\rangle = 0.$$

Then, by the **Z**-property of L on K, we have

$$\langle L(x), y\rangle \leq 0.$$

As $L(x) = y + q$, this leads to $||y||^2 + \langle y, q\rangle \leq 0$. Since $q \in K^\circ$ and $y \in K^*$, we must have $\langle y, q\rangle \geq 0$ and so $y = 0$. This implies that $L(x) = q$. So, for each $q \in K^\circ$, we have an $x \in K$ such that $L(x) = q$. This implies that the range of L contains the open set K° and so L must be onto. As V is finite dimensional, (the linear transformation) L must be one-to-one proving the existence of L^{-1}. We also see that for each $q \in K^\circ$, $L^{-1}(q) = x \in K$. From this we get $L^{-1}(K^\circ) \subseteq K^\circ$ and $L^{-1}(K) \subseteq K$.
$(b) \Rightarrow (c)$: Assuming (b), let $q \in K^\circ$ so that $L^{-1}(q) = d \in K^\circ$. Then $d \in K^\circ$ with $L(d) \in K^\circ$, proving (c).

$(c) \Rightarrow (d)$: Let d be as in (c). We show that for $t = 0$ or 1, the problem LCP(L^T, K^*, td) has only one solution, namely, zero. Note that zero is a solution of this problem; let v be any solution so that

$$v \in K^*, \ \ w = L^T(v) + td \in K, \quad \text{and} \quad \langle v, w \rangle = 0.$$

Then, by the **Z**-property of L on K, we have

$$\langle L(w), v \rangle \leq 0.$$

As $\langle L(w), v \rangle = \langle w, L^T(v) \rangle$, we get

$$0 \geq \langle L^T(v) + td, L^T(v) \rangle = ||L^T(v)||^2 + t\langle L(d), v \rangle.$$

Since $L(d) \in K^\circ$, $v \in K^*$, and $t \geq 0$, we must have $L^T(v) = 0$ which further yields

$$\langle L(d), v \rangle = \langle L^T(v), d \rangle = 0.$$

As $L(d) \in K^\circ$, $v \in K^*$, this can happen only if $v = 0$. So zero is the only solution of LCP(L^T, K^*, td) for $t = 0$ or 1. Since d is in the interior of $(K^*)^* = K$, by Theorem 2, we get (d).
$(d) \Rightarrow (e)$: Since L^T has the **Z**-property with respect to K^*, the proof of $(d) \Rightarrow (e)$ is similar to that of $(a) \Rightarrow (b)$.
$(e) \Rightarrow (f)$: This is similar to $(b) \Rightarrow (c)$.
$(f) \Rightarrow (a)$: The proof is similar to that of $(c) \Rightarrow (d)$: We work with LCP(L, K, tu) and show that for $t = 0, 1$, this problem has a unique solution, namely, zero, and then appeal to Theorem 2. □

Remark 5 We note that (only) the nonemptyness of K° was used in proving the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ and (only) the nonemptyness of $(K^*)^o$ was used in the proofs of $(d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (a)$.

The above result shows that with respect to K,

$$\mathbf{Z} \cap \mathbf{S} = \mathbf{Z} \cap \mathbf{Q}.$$

Analogous to the implication $(a) \Rightarrow (c)$, we have the following for any L: If $L \in \mathbf{Q}(K)$, then there exists a $p \in K^\circ$ such that $L(p) \in (K^*)^o$. This can be seen by taking an $e \in (K^*)^o$, looking at a solution x of LCP(L, K, $-e$) (so that $L(x) - e \in K^*$, that is, $L(x) \in (K^*)^o$), and finally perturbing x.

Remark 6 The equivalence of (b) and (c) for transformations of the form $L = R - S$ where R and S satisfy the conditions $S(K) \subseteq K$ and $R(K^\circ) \supseteq K^\circ$ or $R(K^\circ) \cap K^\circ = \emptyset$, has been studied by Schneider [30]. In fact, he proves the equivalence of the following:

(i) R is invertible, $R^{-1}(K) \subseteq K$, and $\rho(R^{-1}S) < 1$.
(ii) L is invertible and $L^{-1}(K^\circ) \subseteq K^\circ$.
(iii) There exists a vector $d \in K^\circ$ such that $L(d) \in K^\circ$.

We note that such transformations need not imply the **Q**-property (cf. Example 2, [14]) and so need not have the **Z**-property. In [14], Gowda and Parthasarathy show that when R is a multiple of the Identity, the above conditions are further equivalent to the **Q**-property of L. They also observe (cf. Example 2, [14]) that the **Q**-property does not hold for a general L.

The following is an example for which both Theorem 6 and Schneider's result hold: On $\mathcal{S}^n$ with $K = \mathcal{S}^n_+$, consider a transformation of the form

$$L(X) = AX + XA^T - \sum_1^k B_i X B_i^T.$$

Then L, being a sum of **Z**-transformations, is a **Z**-transformation. If there is a $D \succ 0$ such that $L(D) \succ 0$, then equivalent conditions in Theorem 6 hold. Also, writing $L = R - S$ with $R(X) = AX + XA^T$ (which is a **Z**-transformation), we see that $R(D) \succ 0$ and so $R^{-1}(K^\circ) \subseteq K^\circ$ where K is the interior of $\mathcal{S}^n_+$; Schneider's result now applies.

The following result describes the eigenvalues of $\mathbf{Z} \cap \mathbf{S}$-transformations.

Theorem 7 *Let K be proper and suppose $L \in \mathbf{Z}(K)$. Then the following are equivalent*:

(i) *L is positive stable.*
(ii) *All real eigenvalues of L are positive.*
(iii) *$L + \varepsilon I$ is nonsingular for all $\varepsilon \geq 0$.*
(iv) *L has the **Q**-property with respect to K.*

Proof The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. Now suppose (iii) holds. Define the function $F_0 : V \to V$ by

$$F_0(x) := x - \Pi_K(x - L(x)).$$

We show that for some bounded open set Ω containing zero in V, $deg(F_0, \Omega, 0) = 1$. Then Theorem 1 shows that $L \in \mathbf{Q}(K)$. Now define the homotopy $H(x, t) : V \times [0, 1] \to V$ by

$$H(x, t) := x - \Pi_K(x - (1 - t)L(x) - tx).$$

Then $H(x, 0) = F_0(x)$ and $H(x, 1) = Ix := x$ for all $x \in V$. We claim that as t varies from 0 to 1, the zero sets of H are uniformly bounded. Assuming the contrary, we have sequences x_n and t_n such that $||x_n|| \to \infty$ and $t_n \in [0, 1]$ with $H(x_n, t_n) = 0$. Then $x_n \in K$, $y_n := (1 - t_n)L(x_n) + t_n x_n \in K^*$, and $\langle x_n, y_n \rangle = 0$. Assuming $\frac{x_n}{||x_n||} \to \overline{x}$ and $t_n \to \overline{t} \in [0, 1]$, we have

$$\overline{x} \in K, \ \overline{y} = (1 - \overline{t})L(\overline{x}) + \overline{t}\overline{x} \in K^*, \ \text{and} \ \langle \overline{x}, \overline{y} \rangle = 0.$$

As $||\overline{x}|| = 1$, $\overline{t} \neq 1$. Then $\overline{x}$ is a solution of LCP$(L + \overline{s}I, K, 0)$ where $\overline{s} = \frac{\overline{t}}{1-\overline{t}}$. Observe that $L + \overline{s}I$ belongs to $\mathbf{Z}(K)$ and so

$$||(L + \overline{s}I)\overline{x}||^2 = \left\langle (L + \overline{s}I)\overline{x}, \left(\frac{1}{1-\overline{t}}\right)\overline{y} \right\rangle \leq 0.$$

This gives $(L + \overline{s}I)\overline{x} = 0$ contradicting (iii). This contradiction shows that the zero sets of H are uniformly bounded. Let Ω be any bounded open set containing all the zero sets. Then by the homotopy invariance of degree,

$$\deg(F_0, \Omega, 0) = \deg(I, \Omega, 0) = 1.$$

Now by Theorem 1, L has the **Q**-property with respect to K.

Now suppose (iv) holds. As L belongs to $\mathbf{Z}(K)$, with $\sigma(L)$ denoting the spectrum of L, Theorem 6 in [31] shows that

$$\lambda = \min\{Re(\mu) : \mu \in \sigma(L)\}$$

is an eigenvalue of L with a corresponding eigenvector u in K. Since L is invertible (see Theorem 6), $\lambda \neq 0$. Then $L(u) = \lambda u$ implies that $L^{-1}(u) = \frac{1}{\lambda}u$. Since $L^{-1}(K) \subseteq K$ (see Theorem 6), we must have $\lambda > 0$. This proves that L is positive stable. □

Remark 7 The equivalence of items (b), (c), (e), (f) in Theorem 6 and (i) in Theorem 7 has also been noted by Stern [32] by different means. See also, Chap. 5 in [2] and [34].

In addition, Theorem 4.3 in [34] says that

$$\mathbf{Z}(K) \cap \mathcal{S}(K) = \{L : (L + \varepsilon I)^{-1}(K) \subseteq K \quad \forall \quad \varepsilon \geq 0\}.$$

By the well known Lyapunov theory, the positive stability of a transformation L is equivalent to the global asymptotic stability of

$$\frac{dx}{dt} + L(x) = 0$$

(that is, its trajectory, starting at any point in V, converges to zero). As remarked earlier, the **Z**-property of L with respect to K is equivalent to the viability of the above dynamical system in K (which means the trajectory, starting at any point in K, stays in K) or the so-called exponential nonnegativity of $-L$ (Chap. 4, [2]).

Corollary 1 *Let K be proper and suppose $L \in \mathbf{Z}(K) \cap \mathbf{S}(K)$. Then*

(a) *The determinant of L is positive*;
(b) *If L is self-adjoint, then L is positive definite.*

Proof From the previous theorem, L is positive stable. As L is a real linear transformation, the complex eigenvalues of L occur in conjugate pairs. Thus the determinant of L, being the product of all its eigenvalues, is positive. If L is self-adjoint on V, then it is so on the complexification of V. Thus all eigenvalues of L are real and positive. Therefore L is positive definite on the complexification of V and hence on V. □

Remark 8 Let $V = R^n$ and $K = R^n_+$. Let M be a **Z**-matrix such that for some $d > 0$, we have $Md > 0$. It follows that for every nonempty $\alpha \subseteq \{1, 2, \ldots, n\}$, the principal submatrix $M_{\alpha\alpha}$ is a **Z**-matrix with $M_{\alpha\alpha} d_\alpha > 0$. Then the above corollary implies that the determinant of $M_{\alpha\alpha}$ is positive. Thus we have the well-known result that every principal minor of a $\mathbf{Z} \cap \mathbf{S}$-matrix is positive, i.e., every $\mathbf{Z} \cap \mathbf{S}$-matrix is a **P**-matrix. This result cannot be extended to general cones, as the following example shows.

Example 7 Consider, on $\mathcal{S}^2$, L_A with

$$A = \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}.$$

We know that L_A has the **Z**-property on $\mathcal{S}^2_+$. As A is positive stable, L_A has the **Q**-property on $\mathcal{S}^2_+$ [15]. Now consider the principal subtransformation of L_A corresponding to the face F consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$$

where x is nonnegative. Then this principal subtransformation is given, on $Span(F)$, by

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} -2x & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly, the determinant of this subtransformation is negative and so the positive principal minor property (see Sect. 2) does not hold for the $\mathbf{Z} \cap \mathbf{S}$-transformation L_A.

4 Simultaneous stability, sum and product Results

Given a set $\mathcal{A}$ of matrices, the simultaneous stability problem is: Find a (symmetric) positive definite matrix X such that $AX + XA^T$ is positive definite for all $A \in \mathcal{A}$. As is well known, this stability problem is related to the asymptotic stability of the linear time-varying system

$$\frac{dx}{dt} = -A(t)x$$

where $x(t) \in R^n$ and $A(t) \in \mathcal{A}$ for all t (Sect. 5.1, [5] or [25]).

In connection with this problem, Narendra and Balakrishnan [27] prove the following

Theorem 8 *Let* $\{A_1, \ldots, A_k\}$ *consist of (pairwise) commuting positive stable matrices. Then there exists* $X \succ 0$ *such that* $L_{A_i}(X) := A_i X + X A_i^T \succ 0$ *for all* $i = 1, \ldots, k$.

We now extend this result to **Z**-transformations as follows:

Theorem 9 *Suppose* K *is proper and* $L_i \in \mathbf{Z}(K) \cap \mathbf{S}(K)$ *for* $i = 1, 2, \ldots, k$. *If* L_i*s commute pairwise, then there exists a* $d \in K^\circ$ *such that* $L_i(d) \in K^\circ$ *for all* $i = 1, 2 \ldots, k$; *Moreover,* $\sum_1^k L_i \in \mathbf{Q}(K)$.

Proof Suppose that L_is commute pairwise. Since $L_i^{-1}(K^\circ) \subseteq K^\circ$ for all i, $\Pi_1^k L_i^{-1}(K^\circ) \subseteq K^\circ$. Take $u \in K^\circ$ so that $d := \Pi_1^k L_i^{-1}(u) \in K^\circ$. Then using the commutativity of L_is, for any j, $L_j(d) = \Pi_{i \neq j} L_i^{-1}(u) \in K^\circ$. The second conclusion follows from Theorem 6. □

Corollary 2 *Let* $L, L^T \in \mathbf{Z}(K)$. *If there is a* $d \in K^\circ$ *with* $L(d) \in K^\circ$ *and* $L^T(d) \in K^\circ$, *then* L *is positive definite. In particular, this conclusion holds if* L *is normal (that is,* $LL^T = L^T L$).

Proof Since L and L^T are in $\mathbf{Z}(K)$, $L + L^T \in \mathbf{Z}(K)$. Since $(L + L^T)(d) \in K^\circ$ by our assumption, $L + L^T$ is positive definite by Corollary 1. For the second part, we apply the previous theorem. □

Remark 9 When K is self-dual and $L \in \mathbf{Z}(K)$, we have $L^T \in \mathbf{Z}(K)$. So in this setting, the above corollary says that normal $\mathbf{Z} \cap \mathbf{S}$-transformations are positive definite. In particular, when the above corollary is specialized to L_A and S_A, we recover previously proved results (Theorems 2, 3 in [17]).

The following result describes the **Q**-property of a finite product of transformations each belonging to $\mathbf{Z} \cap \mathbf{Q}$ when K is proper and $K^* \subseteq K$. Such a result for Lyapunov and Stein transformations on the semidefinite cone was proved in [16] using degree theoretic ideas. For some recent results of this type on self-dual cones, see [36].

Theorem 10 *Suppose that* K *is proper,* $K^* \subseteq K$, *and* $L_i \in \mathbf{Z}(K) \cap \mathbf{Q}(K)$ *for* $i = 1, \ldots, n$. *Then* $\Pi_1^n L_i$ *has* **Q***-property with respect to* K.

Proof Since $L_i \in \mathbf{Z}(K) \cap \mathbf{Q}(K)$, by Theorem 6, $(L_i^T)^{-1}$ exists and $(L_i^T)^{-1}(K^*) \subseteq K^*$ for all i. Letting $L = \Pi_1^n L_i$, we get $(L^T)^{-1}(K^*) \subseteq K^* \subseteq K$. We now claim that

(a) L^{-1} is copositive on K^*, and
(b) L^{-1} has the $\mathbf{R}_0$-property with respect to K^*.

To see (a), let $x \in K^*$. Then $y := (L^T)^{-1}(x) \in K^* \subseteq K$. Hence

$$\langle L^{-1}(x), x\rangle = \langle x, (L^T)^{-1}(x)\rangle = \langle x, y\rangle \geq 0.$$

To see (b), let $x \in K^*$ with $L^{-1}(x) \in K$ and $\langle L^{-1}(x), x\rangle = 0$. Since $L_n \in \mathbf{Z}(K)$, we have $\langle x, L_n L^{-1}(x)\rangle \leq 0$. But $(L_n L^{-1})^T(x) = (\Pi_1^{n-1}(L_i^T)^{-1})(x) \in K^* \subseteq K$ and hence $\langle x, (L_n L^{-1})^T(x)\rangle \geq 0$. Thus, $\langle x, L_n L^{-1}(x)\rangle = 0$. Now $L_n L^{-1} =$

$(\Pi_1^{n-1} L_i)^{-1}$. By repeating this argument n times, we get $\langle x, x\rangle = 0$, that is, $x = 0$, proving item (b).

As a consequence of Karamardian's Theorem (see Sect. 2), L^{-1} has the **Q**-property with respect to K^*, or equivalently, L has the **Q**-property with respect to K. □

Remark 10 We note that the above theorem is applicable to self-dual cones, and in particular to symmetric cones.

5 Diagonal stability in symmetric cones

In the rest of the paper, we assume that V is a Euclidean Jordan algebra and K is the corresponding symmetric cone. From now on, all the properties are defined with respect to this (fixed) K. We use the standard notations

$$x \geq 0 \quad \text{and} \quad x > 0$$

to mean $x \in K$ and $x \in K^\circ$ respectively. Also, $y \leq 0$ means that $-y \geq 0$.

With respect to K, the **Z**-property can be characterized as follows:

*L has the **Z**-property if and only if for any Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V and $i \neq j$, it holds that $\langle L(e_i), e_j\rangle \leq 0$.*

(If $x \geq 0$, $y \geq 0$, and $\langle x, y\rangle = 0$, then there is a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ such that $x = \sum \lambda_i e_i$, $y = \sum \mu_i e_i$ with $\lambda_i, \mu_i \geq 0$ and $\lambda_i \mu_i = 0$ for all i. This leads to the justification of the "if" part. The "only if" part is obvious.)

Following [11], for any $a \in V$, we define the Lyapunov transformation L_a and quadratic representation P_a by

$$L_a(x) := a \circ x \quad \text{and} \quad P_a = 2(L_a)^2 - L_{a^2}.$$

Because $\langle L_a(x), y\rangle = \langle a \circ x, y\rangle = \langle a, x \circ y\rangle$, we immediately notice (via Proposition 1) that L_a has the **Z**-property with respect to K. Some useful properties are [11]:

- Transformations L_a and P_a are self-adjoint on V.
- When $a > 0$, both L_a and P_a are positive definite on V.
- When a is invertible, $P_a(K) = K$ and $(P_a)^{-1} = P_{a^{-1}}$.
- $P_{P_a(x)} = P_a P_x P_a$; in particular, $P_{a^2} = (P_a)^2$.

The following result shows that any element of K° can be mapped to any element of K° by Lyapunov and quadratic representations. The result is known. The first part is an application of Theorem 6 (see [28] for a different proof) and the second part is crucially needed in our next Theorem.

Proposition 2 *Suppose $u, v > 0$ in V. Then there exist $w > 0$ and $a > 0$ such that*

$$L_w(u) = v \quad \textit{and} \quad P_a(u) = v.$$

Proof Consider the Lyapunov transformation L_u. This is positive definite on V and hence has the **Q**-property with respect to K. Since it also has the **Z**-property, by Theorem 6, $(L_u)^{-1}(K^\circ) \subseteq K^\circ$. Let $(L_u)^{-1}(v) = w$. Then $w > 0$, $L_w(u) = L_u(w) = v$.

As to the existence of a, let

$$a := u^{-1} \# v := P_{u^{-\frac{1}{2}}} \left(P_{u^{\frac{1}{2}}}(v) \right)^{\frac{1}{2}}.$$

(The right-hand side is the so-called geometric mean of u^{-1} and v.) By Theorem 4 in [23], we have

$$a > 0 \quad \text{and} \quad P_a(u) = v.$$

□

For a matrix $A \in R^{n \times n}$, the famous Lyapunov's theorem ([20], Theorem 2.2.1) asserts that A is positive stable if and only if there exists a symmetric positive definite matrix $P \in R^{n \times n}$ such that $AP + PA^T$ is positive definite. In this situation, the dynamical system $\frac{dx}{dt} = -Ax$ is globally asymptotically stable and the Lyapunov function given by $f(x) = \langle Px, x \rangle$ decreases along any trajectory of this system. If A is positive stable and a **Z**-matrix, then, in addition to the global asymptotic stability, we have viability of the system $\frac{dx}{dt} = -Ax$ in R^n_+ (this means that any trajectory that starts at a point in R^n_+ stays in R^n_+) and A is diagonally stable (that is there is a P which is diagonal). In the case of the Euclidean Jordan algebra $V = R^n$ (see Remark), we may think of any diagonal matrix as either L_a or P_a for some $a \in R^n$. Also, in this setting, if $a > 0$, then $P_a = L_{a^2}$; so a matrix A is diagonally stable if there is an $a > 0$ in R^n such that $AP_a + P_aA^T = AL_{a^2} + L_{a^2}A^T$ is positive definite. With this in mind, we generalize Item (6) of the Introduction in the following way:

Theorem 11 *For $L \in \mathbf{Z}$ on a Euclidean Jordan algebra V, the following are equivalent*:

(i) $L \in \mathbf{S}$.
(ii) *There exists $a > 0$ such that $LP_a + P_aL^T$ is positive definite on V.*

Proof Assume (i) holds. Then by Theorem 6, there exists $u > 0$ and $v > 0$ such that $L^T(u) > 0$ and $L(v) > 0$. Then by the previous proposition, there exists an a such that

$$a > 0 \quad \text{and} \quad P_a(u) = v.$$

Put $b = a^{\frac{1}{2}}$. As K is invariant under P_b and $P_{b^{-1}} = (P_b)^{-1}$, it easily follows (from $L \in \mathbf{Z}$) that $P_{b^{-1}}LP_b \in \mathbf{Z}$; hence (by the self-duality of K),

$$\Gamma := P_{b^{-1}}LP_b + [P_{b^{-1}}LP_b]^T \in \mathbf{Z}$$

and $P_{b^{-1}}\Gamma P_b \in \mathbf{Z}$. Letting $\Lambda := P_{b^{-1}}\Gamma P_b$, we see that $\Lambda = P_{a^{-1}}LP_a + L^T \in \mathbf{Z}$. Then

$$\Lambda(u) = P_{a^{-1}}L(P_a(u)) + L^T(u) = P_{a^{-1}}L(v) + L^T(u) > 0$$

as $P_{a^{-1}}$ keeps the interior of K invariant. This implies

$$P_{b^{-1}}\Gamma P_b(u) > 0 \quad \text{and} \quad \Gamma P_b(u) > 0.$$

So we have proved that Γ (which is symmetric) is in **Z** and has the **S**-property. By Corollary 1, Γ is positive definite; it follows that $P_b\Gamma P_b$ is also positive definite. Since $P_b\Gamma P_b = LP_a + P_aL^T$, we get (ii).

Now suppose (ii) holds. If there is no $d > 0$ such that $L(d) > 0$, then by a Theorem of Alternative, see Page 9 in [3], there is a nonzero y such that

$$y \geq 0 \quad \text{and} \quad L^T(y) \leq 0.$$

As $P_a(y) \geq 0$, we have

$$0 < \langle (LP_a + P_aL^T)(y), y\rangle = 2\langle LP_a(y), y\rangle = 2\langle P_a(y), L^T(y)\rangle \leq 0,$$

which is a contradiction. Hence (i) holds. □

Remark 11 In the above result, we can replace (ii) by: There exists a $c > 0$ such that $P_cL + L^TP_c$ is positive definite on V. This is because, when $L \in \mathbf{Z}$, conditions $L \in \mathbf{S}$ and $L^T \in \mathbf{S}$ are equivalent, see Theorem 6.

6 The P-Property in Euclidean Jordan algebras

Recall that a real $n \times n$ matrix M is a **P**-matrix if all its principal minors are positive. This property can be equivalently described by

$$x * (Mx) \leq 0 \Rightarrow x = 0$$

where $x * (Mx)$ denotes the componentwise product and the inequality is componentwise. This property can be extended to a Euclidean Jordan algebra as follows [18].

Definition 2 Let V be a Euclidean Jordan algebra. A linear transformation $L : V \rightarrow V$ is said to have the **P**-property if

$$\left.\begin{array}{r} x \circ L(x) \leq 0 \\ x \quad \text{operator commutes with} \quad L(x) \end{array}\right\} \Rightarrow x = 0.$$

Note: The (stronger) Jordan **P**-property is obtained by deleting the operator commutativity condition in the above definition.

While the **P**-property always implies the **Q**-property, it does not give uniqueness of solution in related linear complementarity problems over K. However, the **P**-property is implied by the GUS-property (which means that all related linear complementarity problems over K have unique solutions), order **P**-property, Jordan **P**-property, see [18]. Also, the **P**-property does not imply the so-called positive principal minor property, see [18].

Theorem 12 *Let $L : V \to V$ be linear and suppose there exists a $b > 0$ such that $L_b L + L^T L_b$ is positive definite on V. Then L has the* **P***-property.*

Proof Suppose $x \neq 0$ and $x \circ L(x) \leq 0$. Then

$$0 < \langle (L_b L + L^T L_b)(x), x \rangle = 2\langle L_b(x), L(x) \rangle = 2\langle b, x \circ L(x) \rangle \leq 0$$

as $b > 0$ and $x \circ L(x) \leq 0$. It follows that $x \circ L(x) \leq 0$ implies $x = 0$ proving the Jordan **P**-property of L. Thus we have the **P**-property of L. □

As the **P**-property always implies the **Q**-property (hence the **S**-property) [18], combining the above theorem with Theorem 11, we get the following

Corollary 3 *If $L \in \mathbf{Z}$ and there exists a $b > 0$ such that $L_b L + L^T L_b$ is positive definite, then there exist an $a > 0$ such that $L P_a + P_a L^T$ is positive definite.*

7 On the equality $\mathbf{Z} \cap \mathbf{Q} = \mathbf{Z} \cap \mathbf{P}$ in Euclidean Jordan algebras

As we saw in the Introduction, for a **Z**-matrix, the **Q** and **P** properties coincide. We ask if the same is true in the context of a Euclidean Jordan algebra. Since the **P**-property implies the **Q**-property for any linear transformation, we ask if the inclusion

$$\mathbf{Z} \cap \mathbf{Q} \subseteq \mathbf{P}$$

holds. We begin with some examples.

Example 8 Consider L_A (defined in Example 2) on $\mathcal{S}^n$. We have observed previously that L_A has the **Z**-property. The equivalence of **Q** and **P** properties has been established in Theorem 5 of [15].

Example 9 Consider S_A (defined in Example 3) on $\mathcal{S}^n$. We have observed previously that S_A has the **Z**-property. The equivalence of **Q** and **P** properties has been established in Theorem 11 of [14].

Example 10 Let V be any Euclidean Jordan algebra. For any $A \in R^{r \times r}$ and any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, consider R_A. Suppose that $R_A \in \mathbf{Z} \cap \mathbf{Q} = \mathbf{Z} \cap \mathbf{S}$. Let $d > 0$ with $R_A(d) > 0$. Then using the Peirce decomposition of d with respect to $\{e_1, e_2, \ldots, e_r\}$, we see that $A \in \mathbf{S}(R^n_+)$. As A is a **Z**-matrix, see Example 5, A is a $\mathbf{Z} \cap \mathbf{Q}$-matrix. Hence it is a **P**-matrix. It follows, see Proposition 5.1 in [38], that R_A has the **P**-property.

Regarding the equality $\mathbf{Z} \cap \mathbf{Q} = \mathbf{Z} \cap \mathbf{P}$, we do not have an answer in the general case. (For $L \in \mathbf{Z} \cap \mathbf{Q}$, we know (by Remark 11) that there exists $c > 0$ such that $P_c L + L^T P_c$ is positive definite. If one can show that P_c can be replaced by some L_b, then it follows from Theorem 12 that $L \in \mathbf{P}$.) The recent article [36] contains some partial answers in the case of $L = I - S$ with $S(K) \subseteq K$. For $\mathcal{L}^n$, we have the following result.

Theorem 13 *Let $V = \mathcal{L}^n$. Then $\mathbf{Z} \cap \mathbf{Q} = \mathbf{Z} \cap \mathbf{P}$.*

Proof Let $L : \mathcal{L}^n \to \mathcal{L}^n$ be in $\mathbf{Z} \cap \mathbf{Q} = \mathbf{Z} \cap \mathbf{S}$. We have to show that L has the $\mathbf{P}$-property. To this end, let $x \neq 0$ and $L(x)$ operator commute in $\mathcal{L}^n$ with $x \circ L(x) \leq 0$. Because of operator commutativity, there is a Jordan frame $\{e_1, e_2\}$ in $\mathcal{L}^n$ such that

$$x = \lambda_1 e_1 + \lambda_2 e_2 \quad \text{and} \quad L(x) = \mu_1 e_1 + \mu_2 e_2.$$

Now $x \circ L(x) \leq 0$ yields $\sum \lambda_i \mu_i e_i \leq 0$. It follows that $\lambda_i \mu_i \leq 0$ for $i = 1, 2$.

We consider several cases.

Case 1 $\lambda_1 \neq 0, \lambda_2 = 0$ (Note: The case $\lambda_1 = 0, \lambda_2 \neq 0$ is similar.) Then $x = \lambda_1 e_1 \Rightarrow \frac{x}{\lambda_1} = e_1, L(\frac{x}{\lambda_1}) = \frac{\mu_1}{\lambda_1} e_1 + \frac{\mu_1}{\lambda_2} e_2 \Rightarrow L(e_1) = r_1 e_1 + r_2 e_2$, where $r_i = \frac{\mu_i}{\lambda_1}$ for $i = 1, 2$. As $\lambda_1 \mu_1 \leq 0$ we have $r_1 \leq 0$. Since $e_1 \geq 0, e_2 \geq 0$ and $\langle e_1, e_2 \rangle = 0$, from the $\mathbf{Z}$-property, we have $\langle L(e_1), e_2 \rangle \leq 0$; thus $r_2 \leq 0$. Hence $L(e_1) = r_1 e_1 + r_2 e_2 \leq 0$. As $L \in \mathbf{Z} \cap \mathbf{Q}$, from Theorem 6, $L^{-1}(\mathcal{L}^n_+) \subseteq \mathcal{L}^n_+$; this implies that $e_1 = L^{-1}(r_1 e_1 + r_2 e_2) \leq 0$ which is clearly a contradiction.

Case 2 $\lambda_1 \neq 0, \lambda_2 \neq 0$.

Subcase 2.1 $\lambda_1 > 0$, $\lambda_2 > 0$. (Note: The case $\lambda_1 < 0$, $\lambda_2 < 0$ can be handled by working with $-x$.) In this case, $x > 0$, $\mu_1 \leq 0$ and $\mu_2 \leq 0$. As $L^{-1}(e_1) \geq 0$ and $L^{-1}(e_2) \geq 0$, we have $L(x) = \mu_1 e_1 + \mu_2 e_2 \Rightarrow x = \mu_1 L^{-1}(e_1) + \mu_2 L^{-1}(e_2) \leq 0$, which is a contradiction.

Subcase 2.2 $\lambda_1 > 0, \lambda_2 < 0$. (Note: The case $\lambda_1 < 0, \lambda_2 > 0$ is similar.) In this case, $\mu_1 \leq 0$ and $\mu_2 \geq 0$. Also, $x \ngeq 0$ and $x \nleq 0$.

If $\mu_1 = 0$, then we have

$$x = \mu_2 L^{-1}(e_2) \Rightarrow \lambda_1 e_1 + \lambda_2 e_2 = \mu_2 L^{-1}(e_2)$$

and

$$0 > \lambda_2 ||e_2||^2 = \mu_2 \langle L^{-1}(e_2), e_2 \rangle \geq 0.$$

This is a contradiction.

If $\mu_1 < 0$, let $\mu_1 = -r_1$, so that $r_1 > 0$. Then

$$L(x) + \frac{r_1}{\lambda_1} x = (\mu_2 + \lambda_2 \frac{r_1}{\lambda_1}) e_2 \Rightarrow x = (L + \frac{r_1}{\lambda_1} I)^{-1} \theta e_2,$$

where $\theta = \mu_2 + \lambda_2 \frac{r_1}{\lambda_1}$.

Since $L \in \mathbf{Z} \cap \mathbf{S}$, we easily verify that $(L + \frac{r_1}{\lambda_1} I) \in \mathbf{Z} \cap \mathbf{S}$; thus $(L + \frac{r_1}{\lambda_1} I)^{-1} e_2 \geq 0$. For any θ this cannot happen as $x \ngeq 0$ and $x \nleq 0$. Thus in all cases, we reach a contradiction, proving the $\mathbf{P}$-property of L. □

As an easy consequence (via Example 3 and Remark 6) we deduce the following result which, in [36], was obtained by different means.

Corollary 4 *Suppose $L = I - S$ where S is linear, $\rho(S) < 1$ and $S(\mathcal{L}^n_+) \subseteq \mathcal{L}^n_+$. Then L has the $\mathbf{P}$-property.*

8 Miscellaneous

In the case of a **Z**-matrix M, it is known that feasibility of a (standard) linear complementarity problem implies its solvability. This means that if there is an $x \geq 0$ such that $Mx + q \geq 0$, then LCP(M, R^n_+, q) is solvable. (In the terminology of linear complementarity problems, this says that every **Z**-matrix is a $\mathbf{Q}_0$-matrix.)

The following example shows that an analogous result fails in the general case.

Example 11 Let $V = \mathcal{S}^2$. Corresponding to

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 0 \end{bmatrix},$$

consider LCP$(L_A, \mathcal{S}^2_+, Q)$. Put

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \quad \text{and} \quad Y = L_A(X) + Q = \begin{bmatrix} 2y+1 & x+z+0.1 \\ x+z+0.1 & 2y \end{bmatrix}.$$

Taking $x = z = 1$ and $y = 0.9$, we have $X = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix} \succ 0$, $Y = \begin{bmatrix} 2.8 & 2.1 \\ 2.1 & 1.8 \end{bmatrix} \succ 0$. Thus, LCP$(L_A, Q)$ is (strictly) feasible.

Suppose LCP$(L_A, \mathcal{S}^n_+, Q)$ is solvable, so that there exists $X \succeq 0$ such that $Y \succeq 0$ and $\langle X, Y \rangle = 0$. Now, $X \succeq 0 \Rightarrow x \geq 0$ and $z \geq 0$; $Y \succeq 0 \Rightarrow 2y + 1 \geq 0$ and $y \geq 0$. Also,

$$\langle X, Y \rangle = 0 \Rightarrow x(2y+1) + 2y(x+z+0.1) + 2yz = 0.$$

Because each term in the above equation is nonnegative, we have $x(2y+1) = 0 \Rightarrow x = 0$, $y(x+z+0.1) = 0 \Rightarrow y = 0$, but then

$$Y = \begin{bmatrix} 1 & z+0.1 \\ z+0.1 & 0 \end{bmatrix} \notin S^2_+.$$

Therefore LCP$(L_A, \mathcal{S}^n_+, Q)$ is not solvable.

Concluding remarks and open problems. In this article, we extended the **Z**-matrix property to linear transformations on proper cones. We showed that Lyapunov and Stein transformations have the **Z**-property on the semidefinite cone. We have also shown that many of the properties of **Z**-matrices extend to **Z**-transformations on proper cones. The diagonal stability of such transformations on symmetric cones was described by means of quadratic representations. Finally the equivalence between the **Q** and **P** properties of **Z**-transformations was studied over Euclidean Jordan algebras. Motivated by our study, we formulate the following

Conjecture On any Euclidean Jordan algebra, $\mathbf{Z} \cap \mathbf{Q} = \mathbf{Z} \cap \mathbf{P}$.

In this paper we have not addressed the uniqueness of solutions in cone linear complementarity problems. As is well-known, when $V = R^n$ and $K = R^n_+$, the positive principle minor property (see Sect. 2) gives a characterization of uniqueness of solution in (standard) linear complementarity problems. By replacing R^n_+ by a polyhedral set and using the concept of coherence, Robinson [29] extended this result to variational inequalities. While the uniqueness issue has been settled for some special classes of transformations (such as Lyapunov transformations, see [15]), there is no constructive or verifiable characterization in the general case of (symmetric) cone linear complementarity problems. It may be interesting to study this uniqueness issue for **Z**-transformations on symmetric cones.

References

1. Bapat, R.B., Raghavan, T.E.S.: Nonnegative Matrices and Applications. Cambridge University Press, Cambridge (1997)
2. Berman, A., Neumann, M., Stern, R.J.: Nonnegative Matrices in Dynamical Systems. Wiley, New York (1989)
3. Berman, A., Plemmons, R.J.: Nonnegative Matrices in the Mathematical Sciences. SIAM, Philadelphia (1994)
4. Borwein, J.M., Dempster, M.A.H.: The order linear complementarity problem. Math. Oper. Res. **14**, 534–558 (1989)
5. Boyd, S., El Ghaoui, L., Feron, E., Balakrishnan, V.: Linear Matrix Inequalities in System and Control Theory. SIAM publications, Philadelphia (1994)
6. Cottle, R.W., Pang, J.-S., Stone, R.E.: The Linear Complementarity Problem. Academic, Boston (1992)
7. Cryer, C.W., Dempster, M.A.H.: Equivalence of linear complementarity and linear programs in vector lattice Hilbert spaces. SIAM J. Control Optim. **18**, 76–90 (1980)
8. Damm, T.: Positive groups on H^n are completely positive. Linear Algebra Appl. **393**, 127–137 (2004)
9. Elsner, L.: Quasimonotonie und ungleischungen in halbgeordneten Räumen. Linear Algebra Appl. **8**, 249–261 (1974)
10. Facchinei, F., Pang, J.-S.: Finite Dimensional Variational Inequalities and Complementarity Problems. Springer, New York (2003)
11. Faraut, J., Korányi, A.: Analysis on Symmetric Cones. Oxford University Press, Oxford (1994)
12. Fiedler, M., Pták, V.: On matrices with non-positive off-diagonal elements and positive principle minors. Czechoslov. Math. J. **12**, 382–400 (1962)
13. Finsler, P.: Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratisher Formen. Comment. Math. Helv. **9**, 188–192 (1937)
14. Gowda, M.S., Parthasarathy, T.: Complementarity forms of theorems of Lyapunov and Stein, and related results. Linear Algebra Appl. **320**, 131–144 (2000)
15. Gowda, M.S., Song, Y.: On semidefinite linear complementarity problems. Math. Program. Series A **88**, 575–587 (2000)
16. Gowda, M.S., Song, Y.: Some new results for the semidefinite linear complementarity problem. SIAM J. Matrix Anal. Appl. **24**, 25–39 (2002)
17. Gowda, M.S., Song, Y., Ravindran, G.: On some interconnections between strict monotonicity, globally uniquely solvable, and P properties in semidefinite linear complementarity problems. Linear Algebra Appl. **370**, 355–368 (2003)
18. Gowda, M.S., Sznajder, R., Tao, J.: Some **P**-properties for linear transformations on Euclidean Jordan algebras. Linear Algebra Appl. **393**, 203–232 (2004)
19. Gritzmann, P., Klee, V., Tam, B.-S.: Cross-positive matrices revisited. Linear Algebra Appl. **223/224**, 285–305 (1995)
20. Horn, R.A., Johnson, C.R.: Topics in Matrix Analysis. Cambridge University Press, Cambridge (1991)
21. Kaneko, I.: Linear complementarity problems and characterizations of Minkowski matrices. Linear Algebra Appl. **20**, 113–130 (1978)
22. Karamardian, S.: An existence theorem for the complementarity problem. Optim. Theory Appl. **19**, 227–232 (1976)

23. Lim, Y.: Applications of geometric means on symmetric cones. Math. Ann. **319**, 457–468 (2001)
24. Lloyd, N.G.: Degree Theory. Cambridge University Press, Cambridge (1978)
25. Mason, O., Shorten, R.: The geometry of convex cones associated with the Lyapunov inequality and the common Lyapunov function problem. Electron. J. Linear Algebra **12**, 42–63 (2005)
26. Mesbahi, M., Papavassilopoulos, G.P.: Least elements and the minimal rank matrices, in complementarity problems and variational problems, State of the art. In: Proceedings of the International Conference on Complementarity Problems. SIAM Publications, Philadelphia (1997)
27. Narendra, K.S., Balakrishnan, J.: A common Lyapunov function for stable LTI systems with commuting A-matrices. IEEE Trans. Autom. Control **39**, 2469–2471 (1994)
28. Rangarajan, B.K.: Polynomial convergence of infeasible-interior-point methods over symmetric cones. SIAM J. Optim. **16**, 1211–1229 (2006)
29. Robinson, S.M.: Normal maps induced by linear transformations. Math. Oper. Res. **17**, 691–714 (1992)
30. Schneider, H.: Positive operators and an inertia theorem. Numer. Math. **7**, 11–17 (1965)
31. Schneider, H., Vidyasagar, M.: Cross-positive matrices. SIAM J. Numer. Anal. **7**, 508–519 (1970)
32. Stern, R.J.: Generalized M-matrices. Linear Algebra Appl. **41**, 201–208 (1981)
33. Stern, R.J.: A note on positively invariant cones. Appl. Math. Optim. **9**, 67–72 (1982)
34. Stern, R.J., Tsatsomeros, M.: Extended M-matrices and subtangentiality. Linear Algebra Appl. **97**, 1–11 (1987)
35. Stern, R.J., Wolkowicz, H.: Exponential nonnegativity on the ice-cream cone. SIAM J. Matrix Anal. **12**, 160–165 (1991)
36. Sznajder, R., Gowda, M.S.: The **Q**-property of composite transformations and the **P**-property of Stein-type transformations on self-dual and symmetric cones. Linear Algebra Appl. **416**, 437–451 (2006)
37. Tam, B.-S.: Some results on cross-positive matrices. Linear Algebra Appl. **15**, 173–176 (1976)
38. Tao, J., Gowda, M.S.: Some P-properties for nonlinear transformations on Euclidean Jordan algebras. Math. Oper. Res. **30**, 985–1004 (2005)
39. Uhlig, F.: A recurring theorem about pairs of quadratic forms and extensions: a survey. Linear Algebra Appl. **25**, 219–237 (1979)