Chapter 25
On Common Linear/Quadratic Lyapunov Functions for Switched Linear Systems

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Abstract Using duality and complementarity ideas, and Z-transformations, in this article, we discuss equivalent ways of describing the existence of common linear/quadratic Lyapunov functions for switched linear systems. In particular, we extend a recent result of Mason-Shorten on positive switched system with two constituent linear time-invariant systems to an arbitrary finite system.

25.1 Introduction

Given a finite set of matrices \(\{A_1, A_2, \ldots, A_m\}\) in \(\mathbb{R}^{n \times n}\), the dynamical system
\[
\dot{x} + A_{\sigma}x = 0, \quad \sigma \in \{1, 2, \ldots, m\}, \tag{25.1}
\]
where the switching signal \(\sigma\) is a piecewise constant function from \([0, \infty)\) to \(\{1, 2, \ldots, m\}\), is called a switched (continuous) linear system. Because of numerous applications of such systems, these and their variants have been well studied in the literature, see e.g., the monograph [16] and recent survey article [27]. Given a signal \(\sigma\), a solution to (25.1) is a continuous and piecewise continuously differentiable function \(x(t)\) that satisfies \(\dot{x} + A_{\sigma}x(t) = 0\) for all \(t \in [0, \infty)\) except at the switching instances of \(\sigma\). For (uniform exponential) asymptotic stability of (25.1)
corresponding to arbitrary signals, which means that there exist numbers \( M \geq 1 \) and \( \beta > 0 \) such that
\[
||x(t)|| \leq Me^{-\beta t}||x(0)||
\]
for all \( t \geq 0 \), for all solutions \( x(t) \), and all signals, a sufficient condition is the existence of a common quadratic Lyapunov function (CQLF) \( V(x) := x^TPx \), where \( P \) is a symmetric matrix satisfying the conditions
\[
P > 0 \quad \text{and} \quad A_i^T P + PA_i > 0 \quad \text{for all} \quad i = 1, 2, \ldots, m. \tag{25.2}
\]

The existence of such a \( P \) has been studied by numerous authors under various sufficient conditions, such as commutativity [22], simultaneous triangularization [21], Lie algebraic conditions [17], etc. See also, [25], [26], [15], [24].

Similar to the continuous case, there is a switched (discrete) linear system given by
\[
x(k+1) + A_\sigma x(k) = 0, \quad \sigma \in \{1, 2, \ldots, m\}, \quad k = 1, 2, \ldots \tag{25.3}
\]
These systems have also been well studied, see e.g., [19]. As in the continuous case, the stability can be studied by constructing a CQLF \( V(x) := x^TPx \), where \( P \) is a symmetric matrix satisfying the conditions
\[
P > 0 \quad \text{and} \quad P - A_i^T PA_i > 0 \quad \text{for all} \quad i = 1, 2, \ldots, m. \tag{25.4}
\]

If we restrict the dynamics of the system (25.1) to the nonnegative orthant \( \mathbb{R}_n^+ \) of \( \mathbb{R}_n \), we get a positive switched system [7]. Here we require any trajectory of (25.1) with a starting point in \( \mathbb{R}_n^+ \) to remain in \( \mathbb{R}_n^+ \). This condition requires all matrices \( A_i \) to be \( \mathcal{L} \)-matrices (which are matrices with nonpositive off-diagonal entries). In this setting, a common linear Lyapunov function \( V(x) = x^Td \) can be constructed, where \( d \) is a vector in \( \mathbb{R}_n \) satisfying the conditions
\[
d > 0 \quad \text{and} \quad A_i^T d > 0 \quad \text{for all} \quad i = 1, 2, \ldots, m. \tag{25.5}
\]

Alternatively, for the same positive switched system, a common quadratic copositive Lyapunov function \( V(x) := x^TPx \) can be constructed, where \( P \) is a symmetric matrix satisfying the conditions
\[
P \quad \text{and} \quad A_i^T P + PA_i \quad \text{are strictly copositive on} \quad \mathbb{R}_n^+ \quad \text{for all} \quad i = 1, 2, \ldots, m. \tag{25.6}
\]
(Here, a matrix \( A \) is strictly copositive on \( \mathbb{R}_n^+ \) means that \( x^TAx > 0 \) for all \( 0 \neq x \in \mathbb{R}_n^+ \).)

By considering the cone of (symmetric) positive semidefinite matrices and transformation
\[
L(X) = A^TX + XA,
\]
various authors have used theorems of alternative (or duality theory) to formulate conditions equivalent to (25.2). For example, a result of Kamenetskiy and Pyatnitskiy [14] says that condition (25.2) holds if and only if there do not exist positive semidefinite matrices \( Y_i \) (\( i = 1, 2, \ldots, m \)) with at least one \( Y_i \) nonzero such that
\[
\sum_{i=1}^{m} (A_i Y_i + Y_i A_i^T) = 0.
\]

It is possible to write similar equivalent conditions for (25.4), (25.5), and (25.6). Our first objective in this paper is to present a unified result that covers all these equivalent conditions. This result was motivated by the observation that the underlying transformations in (25.2), (25.4), (25.5), and (25.6), namely, the Lyapunov transformation \( L_A \) defined on the cone of positive semidefinite matrices, the Stein transformation \( S_A \) on the cone of positive semidefinite matrices, the matrix \( A \) on the cone of nonnegative orthant in \( \mathbb{R}^n \), and the Lyapunov transformation \( L_A \) (corresponding to a \( Z \)-matrix) on the cone of symmetric copositive matrices have the following common property:

\[
x \in K, \ y \in K^*, \ \text{and}, \ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0,
\]

where \( K \) denotes a cone with dual \( K^* \).

Transformations satisfying the above property are called \( Z \)-transformations. By using the properties of \( Z \)-transformations [12], we formulate our unifying condition, see Theorem 25.8.

Our second objective is to demonstrate the relevance of complementarity problems in the study of common linear/quadratic Lyapunov functions. Specifically, we show how a complementarity result of Song, Gowda, and Ravindran [28] can be used to extend a recent result of Mason-Shorten on positive switched systems [20].

Here is an outline of this paper. In Section 25.2, we recall necessary matrix theory concepts and describe essential properties of \( Z \)-transformations. Section 25.3 deals with complementarity ideas. Section 25.4 deals with duality ideas and our unifying result covering the existence of linear/quadratic Lyapunov functions. Finally, in Section 25.5 we present our extension of the Mason-Shorten result.

A word about our notation. Throughout this paper, we will use standard matrix theory and complementarity theory notation. For example, we will use \( Z \)-matrices and positive stable matrices (instead of Metzler matrices and Hurwitz matrices), work with positive definite matrices (instead of negative definite matrices), etc. Also, our (dynamical) systems are written in the form \( \dot{x} + Ax = 0 \) instead of \( \dot{x} = Ax \).

25.2 Preliminaries

25.2.1 Matrix Theory Concepts

The space \( \mathbb{R}^n \) carries the usual inner product which is written as either \( \langle x, y \rangle \) or \( x^T y \). In \( \mathbb{R}^n \), we denote the nonnegative orthant by \( \mathbb{R}_+^n \) and its interior by \( (\mathbb{R}_+^n)^\circ \). We write

\[
d \geq 0 \text{ when } d \in \mathbb{R}_+^n \text{ and } d > 0 \text{ when } d \in (\mathbb{R}_+^n)^\circ.
\]

A matrix \( A \in \mathbb{R}^{n \times n} \) is said to be
• a Z-matrix if all its off-diagonal entries are nonpositive. The negative of a Z-matrix is called a Metzler matrix;
• a P-matrix if all its principal minors are positive;
• a copositive matrix (strictly copositive matrix) if \( x^T A x \geq 0 \) (\( > 0 \)) for all \( 0 \neq x \geq 0 \);
• an S-matrix if there is a \( d > 0 \) such that \( A d > 0 \);
• positive stable if the real part of any eigenvalue of \( A \) is positive;
• Schur stable if the absolute value of any eigenvalue of \( A \) is less than one;
• completely positive if there is a nonnegative (rectangular) matrix \( B \) with \( A = B B^T \), or equivalently, \( A \) is a sum of matrices of the form \( xx^T \) with \( x \geq 0 \).

Consider a nonempty set \( C \) of matrices in \( \mathbb{R}^{n \times n} \). A matrix \( A \in \mathbb{R}^{n \times n} \) is called a column representative of \( C \) if for every \( j = 1, 2, \ldots, n \), the \( j \)th column of \( A \) is the \( j \)th column of some matrix in \( C \). Similarly, row representatives of \( C \) are defined.

For \( A_1, A_2, \ldots, A_m, D_1, \ldots, D_m \in \mathbb{R}^{n \times n} \), where each \( D_i \) is a diagonal matrix, the following holds [29]:

\[
\det \left( \sum_{i=1}^m A_i D_i \right) = \sum \det(K) \det(E),
\]

where \( K \) is a column representative of \( \{ A_1, A_2, \ldots, A_m \} \), \( E \) is a column representative of \( \{ D_1, \ldots, D_m \} \), and the same indexed columns are selected to form \( K \) and \( E \). Here the summation is over all column representatives of \( \{ A_1, \ldots, A_m \} \).

25.2.2 Z-Transformations

Throughout this paper, \( H \) denotes a finite dimensional real Hilbert space. We reserve the symbol \( K \) for a proper cone in \( H \), that is, \( K \) is a closed convex cone in \( H \) such that

\[
K \cap (-K) = \{0\} \quad \text{and} \quad K - K = H.
\]

We denote the dual of \( K \) by \( K^* \) and the interior by \( K^o \).

A linear transformation \( L : H \to H \) is said to be a Z-transformation (or said to have the Z-property) on \( K \) if

\[
x \in K, \ y \in K^*, \ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0.
\]

(25.8)

When \( H = \mathbb{R}^n \) and \( K = \mathbb{R}^n_+ \), a Z-transformation is nothing but a Z-matrix. We remark that a Z-transformation is the negative of a "cross-positive" transformation introduced in [23]. It is known [5] that \( L \) has the Z-property on \( K \) if and only if any trajectory of the dynamical system \( \dot{x} + L(x) = 0 \) which starts in \( K \) stays in \( K \).

The above concept is illustrated in the following examples. Examples 25.1 and 25.2 given below appear in [12]. Example 25.3 is new.

Example 25.1. Let \( H = \mathcal{S}^n \), the space of all \( n \times n \) real symmetric matrices with inner product given by \( \langle X, Y \rangle = \text{trace}(XY) \), where the trace of a matrix is the sum...
of its diagonal elements (or the sum of its eigenvalues). Let \( K = S^+_n \), the cone of positive semidefinite matrices in \( S^n \). We use the notation

\[
X \preceq 0 \text{ when } X \in S^+_n \text{ and } X \succ 0 \text{ when } X \in (S^+_n)^c.
\]

For any \( A \in \mathbb{R}^{n \times n} \), let

\[
L_A(X) = \frac{1}{2}(AX + XA^T) \quad \text{and} \quad S_A(X) = X - AXA^T
\]

denote the Lyapunov and Stein transformations corresponding to \( A \). We verify that \( L_A \) and \( S_A \) are \( Z \)-transformations on \( S^+_n \). Since \( S^+_n \) is self-dual and

\[
\langle L_A(X), Y \rangle = \text{trace}(AXY) = 0 \quad \text{and} \quad \langle S_A(X), Y \rangle = \text{trace}(XY) - \text{trace}(AXA^TY) \leq 0,
\]

where the last inequality comes from the facts that \( AXA^T \succeq 0 \) (when \( X \succeq 0 \)) and the inner product of any two elements in \( S^+_n \) is nonnegative. Thus, both \( L_A \) and \( S_A \) are \( Z \)-transformations on \( S^+_n \).

**Example 25.2.** As in Example 25.1, let \( H = S^n \) with \( \langle X, Y \rangle = \text{trace}(XY) \). Let

\[
K = \{ X \in S^n : X \text{ is copositive on } R_+^n \}.
\]

Then \( K \) is a proper with dual

\[
K^* = \{ X \in S^n : X \text{ is completely positive} \},
\]

see [3], Thm. 2.3. Given \( X \in K \) and \( Y = \sum_{i=1}^N y_i^T \in K^* \) (where \( y_i \in R^n_+ \) for all \( i \)) and \( \langle X, Y \rangle = 0 \), we claim that \( XY \) is a nonnegative matrix with zero diagonal. To see this, first observe that \( \sum_{i=1}^N y_i^T Xy_i = \sum_{i=1}^N \text{trace}(Xy_i y_i^T) = \langle X, Y \rangle = 0 \). As \( X \) is copositive, we get \( y_i^T Xy_i = 0 \) for all \( i \); because \( X \) is copositive and symmetric, we have \( Xy_i = 0 \). (This follows from \( \lim_{x \to 0} \frac{1}{2}(tx + y_i)^T X(tx + y_i) \geq 0 \) for all \( x \geq 0 \).) From this, we see that \( XY = \sum_{i=1}^N Xy_i y_i^T \) is a nonnegative matrix. As \( \text{trace}(XY) = 0 \), \( XY \) must have zero diagonal.

Now suppose \( A \in \mathbb{R}^{n \times n} \). We claim that

\( L_A \) is a \( Z \)-transformation on \( K \) if and only if \( A \) is a \( Z \)-matrix.

To see this, suppose that \( L_A \) is a \( Z \)-transformation on \( K \). We show that the \((1, 2)\) and \((2, 1)\) entries of \( A \) are nonpositive. Let \( e_1, e_2, \ldots, e_n \) denote the standard unit coordinate vectors in \( R^n \). Let

\[
X = [x_{ij}], \quad Y = e_1 e_1^T, \quad \text{and} \quad W = e_2 e_2^T,
\]

where all entries of \( X \) are zero except \( x_{12} = x_{21} = 1 \). We see that \( X \in K \), \( Y, W \in K^* \), and \( \langle X, Y \rangle = 0 = \langle X, W \rangle \). A simple computation shows that

\[
\text{trace}(L_A(X)Y) = a_{12} \quad \text{and} \quad \text{trace}(L_A(X)W) = a_{21}.
\]
By the $Z$-property of $L_A$, we must have $a_{12} \leq 0$ and $a_{21} \leq 0$. A similar argument will prove that other off-diagonal entries are also nonpositive. Hence $A$ is a $Z$-matrix.

Now conversely, suppose that $A$ is a $Z$-matrix. Let $X, Y \in K^*$, with $\langle X, Y \rangle = 0$. Then $XY$ and $YX$ are nonnegative matrices with zero diagonals. Then

$$\text{trace}(LA(X)Y) = \text{trace}(AXY) \leq 0.$$ 

This proves that when $A$ is a $Z$-matrix, $L_A$ has the $Z$-property on $K$.

**Example 25.3.** Let $H$ be a Euclidean Jordan algebra with inner product $\langle \cdot, \cdot \rangle$ and Jordan product $\circ$ [6]. Let $K$ denote the cone of squares $\{x \circ x : x \in H\}$. Then $K$ is a self-dual (proper) cone. Examples of Euclidean Jordan algebras include $\mathbb{R}^n$ with the usual inner product and componentwise product (as the Jordan product), the Jordan spin algebra $L_n$ ($n > 1$) whose underlying space is $\mathbb{R}^{n(n-1)}$ with the usual inner product and $(\alpha_0, X) \circ (\gamma_0, Y) = (\alpha_0 \gamma_0 + \langle X, Y \rangle, \alpha_0 Y + \gamma_0 X)$, and matrix algebras $\mathcal{S}_n$ of all $n \times n$ real symmetric matrices, $\mathcal{H}_n$ of all $n \times n$ complex Hermitian matrices, $\mathcal{Q}_n$ of all $n \times n$ quaternion Hermitian matrices, and $\mathcal{O}_3$ of all $3 \times 3$ octonion Hermitian matrices. In the matrix algebras, the Jordan and inner product are given respectively by $X \circ Y := \frac{1}{2}(XY + YX)$ and $\langle X, Y \rangle := \text{Re}(XY)$.

In any Euclidean Jordan algebra, for any element $a$, the so-called Lyapunov transformation $L_a$ is a $Z$-transformation on the cone of squares, where,

$$L_a(x) = a \circ x.$$ 

We now recall some properties of $Z$-transformations [12].

**Theorem 25.4.** Suppose $L$ is a $Z$-transformation on a proper cone $K$ in a Hilbert space $H$. Then the following are equivalent:

1. There exists a $d \in K^*$ such that $L(d) \in K^*$.
2. $L$ is invertible with $L^{-1}(K^*) \subseteq K^*$.
3. $L$ is positive stable, that is, the real part of any eigenvalue of $L$ is positive.
4. $L + tI$ is invertible for all $t \in [0, \infty]$.
5. All real eigenvalues of $L$ are positive.
6. There is an $e \in (K^*)^\circ$ such that $L^T(e) \in (K^*)^\circ$.

Moreover, when $H = \mathbb{R}^n$ and $K = \mathbb{R}_+$, the above properties (for a $Z$-matrix) are further equivalent to

7. $L$ is a $P$-matrix.

**Remark 25.5.** When $A \in \mathbb{R}^{n \times n}$ is a $Z$-matrix, the positive stability of $A$ can be described in more than 50 equivalent ways, see [2]. In particular, we have the equivalence of the following for a $Z$-matrix:

1. $A$ is positive stable.
2. $A$ is a $P$-matrix.
3. There exists a $d > 0$ such that $Ad > 0$ (or $A^Td > 0$).
4. There exists a (diagonal) $D > 0$ in $\mathcal{S}_n$ such that $AD + DA^T > 0$. 

(5) $L_A$ is positive stable.

(Regarding item (5), we note that the eigenvalues of $L_A$ are of the form $\frac{1}{2} (\lambda + \mu)$, where $\lambda$ and $\mu$ are eigenvalues of $A$.) We now add one more condition to this list:

A Z-matrix $A$ is positive stable if and only if there exists a strictly copositive matrix $C$ such that $AC + CA^T$ is strictly copositive.

This follows from the above theorem applied to $L = L_A$ on the cone of symmetric copositive matrices.

### 25.3 Complementarity Ideas

Suppose $K$ is a self-dual cone in $H$. For any two elements $x, y \in H$, let

$$x \sqcap y = x - \Pi_K (x - y),$$

where $\Pi_K$ denotes the (orthogonal) projection of $H$ onto $K$. It is known, see [11], that $\sqcap$ is a commutative, but (in general) non-associative binary operation on $H$. In the case of $H = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$, we use the standard notation $\wedge$ instead of $\sqcap$; we observe that $x \wedge y = \min \{x, y\}$ and that $\wedge$ is associative.

Given a matrix-tuple $A = (A_1, A_2, \ldots, A_m)$ of matrices $A_i$ in $\mathbb{R}^{n \times n}$ and vector-tuple $q = (q_1, q_2, \ldots, q_m)$ of vectors $q_i$ in $\mathbb{R}^n$, the vertical linear complementarity problem [10] $\text{VLCP}(A, q)$ is to find a vector $x \in \mathbb{R}^n$ such that

$$x \sqcap (A_1 x + q_1) \sqcap \cdots \sqcap (A_m x + q_m) = 0.$$

If $-q_i \in (\mathbb{R}^n_+)^n$, then a solution to the above problem satisfies $x \geq 0$ and $A_i x \geq -q_i > 0$ for all $i$; hence a small perturbation of such an $x$ produces a vector $d > 0$ such that $A_i d > 0$ for all $i$.

**Theorem 25.6. ([10], Section 6.3) The $\text{VLCP}(A, q)$ has a unique solution for all $q$ if and only if every row representative of the set $\{A_1, A_2, \ldots, A_m\}$ is a $P$-matrix.**

When $m = 1$, the VLCP reduces to the well known linear complementarity problem [4] that has been extensively studied in the optimization literature.

Now coming to the general case of a self-dual $K$, it can be easily verified that $x \sqcap y \in K \Rightarrow x, y \in K$. Thus, for a finite set of linear transformations $L_i : H \to H$ and vectors $-e_i \in K^c$, if an $x \in H$ satisfies the complementarity problem

$$x \sqcap f(x) = 0,$$

where $f(x)$ is a combination of $L_i(x) - e_i$, $i = 1, 2, \ldots, m$ under the binary operation $\sqcap$ in some order, then we can assert the existence of a $d \in K^c$ such that $L_i(d) \in K^c$. In particular, when $H = \mathcal{S}^n$ and $K = \mathcal{S}^n_+$ and $L_i = L_{A_i}$, we can relate the existence
of a common quadratic Lyapunov function to a solution of a complementarity problem. We refer the reader to [9] and [12] for results highlighting this connection. See also, Section 25.5.

### 25.4 Duality Ideas

In this section, we present our result that generalizes the result of Kamenetskiy and Pyatnitskiy [14] and at the same time unifies several similar results.

We begin with a theorem of alternative.

**Theorem 25.7.** *(Theorem of Alternative [2], Page 9)* Let $H_1$ and $H_2$ be Hilbert spaces with proper cones $K_1 \subseteq H_1$ and $K_2 \subseteq H_2$. For a linear transformation $L : H_1 \to H_2$, consider the following systems:

(i) $L(x) \in (K_2)^\circ$, $x \in (K_1)^\circ$.

(ii) $L^T(y) \in K_1^*$, $0 \neq y \in -K_2^*$.

(iii) $L(x) \in K_2$, $0 \neq x \in K_1$.

(iv) $L^T(y) \in (K_1^*)^\circ$, $y \in -(K_2^*)^\circ$.

Then, exactly one of the systems (i) and (ii) is consistent and exactly one of the systems (iii) and (iv) is consistent.

**Theorem 25.8.** Let $H$ be a finite dimensional real Hilbert space and $K$ be a proper cone in $H$. Let $L_i : H \to H$ be linear for $i = 1, 2, \ldots, m$, where $L_1$ is positive stable and has the $Z$-property on $K$. Consider the following statements:

(1) There exists a $d \in K^\circ$ such that $L_i(d) \in K$ for all $i = 1, \ldots, m$.

(2) There do not exist $x_i \in K^*$, $i = 1, \ldots, m$ with some $x_i$ nonzero such that $\sum_{i=1}^m L_i^T(x_i) = 0$.

(3) There exists a nonzero $d \in K$ such that $L_i(d) \in K$ for all $i = 1, \ldots, m$.

(4) There do not exist $x_i \in (K^*)^\circ$, $i = 1, \ldots, m$ such that $\sum_{i=1}^m L_i^T(x_i) = 0$.

Then

$(1) \iff (2)$ and $(3) \iff (4)$.

**Proof.** Assume that (1) holds but not (2). Then there are $x_1, x_2, \ldots, x_m$ in $K^*$ with at least one $x_i$ nonzero such that $\sum_{i=1}^m L_i^T(x_i) = 0$. Then

$$0 = \sum_{i=1}^m L_i^T(x_i), d = \sum_{i=1}^m \langle x_i, L_i(d) \rangle > 0$$

as $\langle x_i, L_i(d) \rangle \geq 0$ for all $i = 1, 2, 3, \ldots, m$ and $\langle x_i, L_i(d) \rangle > 0$ when $x_i \neq 0$. This contradiction proves that (1) \Rightarrow (2). Now assume the negation of (1). Since $L_1$ is positive stable and has the $Z$-property on $K$, it is invertible, see Theorem 25.4; hence (as $m \neq 1$) we can define $L : H \to H^{m-1}$ by

$$L(x) := (L_2L_1^{-1}(x), \ldots, L_mL_1^{-1}(x)).$$
We now apply Theorem 25.7 with \( K_1 := K \) in \( H \) and \( K_2 := K \times \ldots \times K \) \((m-1)\) times in \( H^{m-1} \). We claim that condition (i) of Theorem 25.7 cannot hold. Assuming the contrary, there is an \( x \in K^\circ \) such that \( L(x) \in (K_2)^\circ = K^\circ \times \cdots \times K^\circ \). This implies that \( L(x) \in K^\circ \) for all \( i = 2, 3, \ldots, m \). Let

\[ d := L_1^{-1}(x). \]

Since \( L_1^{-1}(K^\circ) \subseteq K^\circ \) (by Theorem 25.4), \( d \in K^\circ \) and \( L_1(d) = x \in K^\circ \). From \( L_i(L_1^{-1}(x)) \in K^\circ \), we get \( L_i(d) \in K^\circ \) for \( i = 2, 3, \ldots, m \). This cannot happen as we have assumed the negation of (1). So condition (i) of Theorem 25.7 fails. Therefore, by condition (ii) of Theorem 25.7, we get a \( y \) such that

\[ 0 \neq y \in -(K^\circ \times \cdots \times K^\circ) \quad \text{and} \quad L(y) \in K^\circ. \]

Now let

\[ -y = (x_2, \ldots, x_m) \in K^\circ \times \cdots \times K^\circ. \]

Then for any \( u \in H \) we have

\[
\langle u, L^T(y) \rangle = \langle L(u), y \rangle = -\langle (L_2L_1^{-1}(u), \ldots, L_mL_1^{-1}(u), (x_2, \ldots, x_m)) \rangle
\]

\[
= -\sum_{i=2}^{m} \langle L_iL_1^{-1}(u), x_i \rangle = -\sum_{i=2}^{m} \langle u, L_i^{-T}L_i^T(x_i) \rangle = \left\langle u, -\sum_{i=2}^{m} L_i^{-T}L_i^T(x_i) \right\rangle.
\]

Hence

\[ x_1 := L^T(y) = -\sum_{i=2}^{m} L_i^{-T}L_i^T(x_i). \]

Applying \( L_i^T \) to both sides of the above equation, we get

\[ L_1^T(x_1) + L_2^T(x_2) + \ldots + L_m^T(x_m) = 0. \]

As \( x_1 \in K^\circ \) and \( (x_2, x_3, \ldots, x_m) \) is nonzero, we get the negation of (2). Hence \( \sim (1) \Rightarrow (2) \), or \( (2) \Rightarrow (1) \). This completes the proof of the equivalence (1) \( \iff \) (2).

The proof of (3) \( \iff \) (4) is similar, but one has to use conditions (iii) and (iv) of Theorem 25.7.

We illustrate the theorem by the following examples.

**Example 25.9.** Let \( H = \mathcal{S}^m, K = \mathcal{S}_+^m \), and

\[ L_A(X) = \frac{1}{2} (A^T X + X A) \quad (X \in \mathcal{S}^m), \]

where \( A_i \in \mathbb{R}^{n \times n} \) \((i = 1, 2, \ldots, m)\) are positive stable. As \( L_A \) and \((L_A)^T = L_A^T\) are both \( \mathbb{Z}\)-transformations on \( \mathcal{S}_+^m \) for any \( A \in \mathbb{R}^{n \times n} \), the above theorem gives the equivalence of the following statements [14]:
(i) There exists $D > 0$ in $\mathcal{P}^n$ such that $A_i^T D + D A_i = 2 L A_i^T (D) > 0$ for all $i = 1, 2, \ldots, m$.
(ii) There do not exist $Y_i \geq 0$ $(i = 1, 2, \ldots, m)$ with at least one nonzero $Y_i$ such that $\sum_{i=1}^{m} (A_i Y_i + Y_i A_i^T) = 0$.

We note that the matrix $D$ in item (i) produces a common quadratic Lyapunov function $V(x) := x^T D x$ for the switched linear system $\dot{x} + A_{\sigma} x = 0$.

**Example 25.10.** Let $H = \mathcal{P}^n$, $K = \mathcal{P}^n$, and $S_A(X) = X - A_i X A_i^T$ $(X \in \mathcal{P}^n)$, where $A_i \in R^{n \times n}$ $(i = 1, 2, \ldots, m)$ are Schur stable. Now, $S_A$ and $(S_A)^T = S_{A^T}$ are $Z$-transformations on $\mathcal{P}^n$ for any $A \in R^{n \times n}$, see Example 25.1. Also, the Schur stability of $A_i$ implies that $S_A$ is positive stable for each $i$, see e.g., Theorem 11 in [8] and Theorem 25.4. Hence the above theorem gives the equivalence of the following statements:

(i) There exists $D > 0$ in $\mathcal{P}^n$ such that $D - A_i^T DA_i > 0$ for all $i = 1, 2, \ldots, m$.
(ii) There do not exist $Y_i \geq 0$ $(i = 1, 2, \ldots, m)$ with at least one nonzero $Y_i$ such that $\sum_{i=1}^{m} (Y_i - A_i Y_i A_i^T) = 0$.

The matrix $D$ in item (i) produces a common quadratic Lyapunov function $V(x) := x^T D x$ for the discrete switched system $x(k+1) + A_{\sigma} x(k) = 0$.

**Example 25.11.** Let $H = \mathcal{P}^n$, $K = \mathcal{P}^n$, and $A_i \in R^{n \times n}$ $(i = 1, 2, \ldots, m)$ be positive stable $Z$-matrices. Then the following are equivalent:

(i) There exists $d > 0$ in $R^m$ such that $A_i^T d > 0$ for all $i = 1, 2, \ldots, m$.
(ii) There do not exist vectors $y_i \geq 0$ $(i = 1, 2, \ldots, m)$ with at least one nonzero $y_i$ such that $\sum_{i=1}^{m} A_i y_i = 0$.

In this setting, the vector $d$ in item (i) produces a common linear Lyapunov function $v(x) := x^T d$ for the positive switched system $\dot{x} + A_{\sigma} x = 0$.

**Example 25.12.** Let $H = \mathcal{P}^n$, $K$ be the cone of symmetric copositive matrices and $A_i \in R^{n \times n}$ for $i = 1, 2, \ldots, m$ be positive stable $Z$-matrices. Then the following are equivalent:

(i) There exists a strictly copositive matrix $C$ in $\mathcal{P}^n$ such that $A_i^T C + C A_i$ is strictly copositive for all $i = 1, 2, \ldots, m$.
(ii) There do not exist completely positive matrices $Y_i$ $(i = 1, 2, \ldots, m)$ with at least one nonzero $Y_i$ such that $\sum_{i=1}^{m} (A_i Y_i + Y_i A_i^T) = 0$.

Here, the matrix $C$ in item (i) produces a common copositive quadratic Lyapunov function $v(x) := x^T C x$ for the positive switched system $\dot{x} + A_{\sigma} x = 0$.

We remark that if condition (i) of Example 25.11 holds, then $C := dd^T$ satisfies condition (i) of Example 25.12. However, the converse is false, see Remark 25.20 in Section 25.5.
25.5 Positive Switched Linear Systems

In a recent paper [20], Mason and Shorten prove that for two positive stable \( Z \)-matrices \( A_1 \) and \( A_2 \), the following are equivalent:

(i) There exists a vector \( d > 0 \) in \( \mathbb{R}^n \) such that \( A_1^T d > 0 \) and \( A_2^T d > 0 \).

(ii) Every column representative of \( \{A_1, A_2\} \) is positive stable.

(iii) Every column representative of \( \{A_1, A_2\} \) has positive determinant.

In an earlier work, based on complementarity ideas, Song, Gowda, and Ravindran [28] proved the equivalence of the following for any compact set \( \mathcal{A} \) of positive stable \( Z \)-matrices in \( \mathbb{R}^{n \times n} \):

(1) There exists a \( d > 0 \) such that \( A^T d > 0 \) for all \( A \in \mathcal{A} \).

(2) Every column representative of \( \mathcal{A} \) is a \( P \)-matrix.

In this section, we extend the above Mason-Shorten result to any compact set of matrices and at the same time improve the result of Song, Gowda, and Ravindran by requiring that only the determinant be positive for every column representative of \( \mathcal{A} \).

We recall that for a \( Z \)-matrix (see Remark 25.5), \( P, S \), and positive stability properties are equivalent.

We begin with some lemmas.

**Lemma 25.13.** Suppose \( \mathcal{A} = \{A_1, A_2, \ldots, A_m\} \subseteq \mathbb{R}^{n \times n} \) with \( \det(C) > 0 \) for any column representative \( C \) of \( \mathcal{A} \). Then for any set of positive diagonal matrices \( \{D_1, \ldots, D_m\} \) we have

\[
\det(A_1 D_1 + \cdots + A_m D_m) > 0.
\]

**Proof.** The result follows from the identity (25.7). \( \square \)

**Lemma 25.14.** Suppose \( A_1, A_2, \ldots, A_m \) are positive stable \( Z \)-matrices and \( \det(C) > 0 \) for any column representative \( C \) of \( \{A_1, A_2, \ldots, A_m\} \). Then there exists \( 0 \neq u \geq 0 \) such that \( A_i^T u \geq 0 \) for all \( i = 1, \ldots, m \).

**Proof.** If the stated conclusion fails, then by the equivalence of items (3) and (4) in Theorem 25.8 (see Example 25.11), there exist \( y_i > 0 \) \( (i = 1, \ldots, m) \) such that

\[
A_1 y_1 + A_2 y_2 + \cdots + A_m y_m = 0.
\]

If \( D_i \) denotes the diagonal matrix with diagonal \( y_i \), then

\[
(A_1 D_1 + A_2 D_2 + \cdots + A_m D_m) e = 0,
\]

where \( e \) is the transpose of the vector \((1, 1, \ldots, 1)\) in \( \mathbb{R}^n \). This means that

\[
\det(A_1 D_1 + A_2 D_2 + \cdots + A_m D_m) = 0.
\]

However, Lemma 25.13 together with the hypothesis shows that this is not possible. \( \square \)
Lemma 25.15. Suppose that $A_1, A_2, \ldots, A_m$ are positive stable $\mathbf{Z}$-matrices and $\det(C) > 0$ for any column representative $C$ of $\{A_1, A_2, \ldots, A_m\}$. Then there exists $d > 0$ such that $A_i^T d > 0$ for all $i = 1, \ldots, m$.

Proof. Let $E$ be the matrix in $\mathbb{R}^{n \times n}$ with every entry 1. We can choose a small positive $\epsilon$ such that $A_i - \epsilon E$ is a $\mathbf{P} \cap \mathbf{Z}$-matrix for all $i$ and any column representative of $\{A_1 - \epsilon E, A_2 - \epsilon E, \ldots, A_m - \epsilon E\}$ has positive determinant. An application of the previous lemma produces a nonzero $u \geq 0$ such that $(A_i - \epsilon E) u \geq 0$ for all $i$. This implies that $0 \neq u \geq 0$ and $A_i^T u > 0$ for all $i$; by continuity, there exists a $d > 0$ such that $A_i^T d > 0$ for all $i$. \qed

For a set $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ in $\mathbb{R}^{n \times n}$, let

$$\mathcal{A}^\# := \text{set of all column representatives of } \mathcal{A}$$

and

$$\mathcal{A} := \left\{ \sum_{i=1}^m A_i D_i : D_i \text{ is a nonnegative diagonal matrix and } \sum_{i=1}^m D_i = I \right\}.$$ 

We note that $\mathcal{A} \subseteq \mathcal{A}^\# \subseteq \mathcal{A}$ and that $\mathcal{A}$ is convex.

Theorem 25.16. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a set of positive stable $\mathbf{Z}$-matrices in $\mathbb{R}^{n \times n}$. Then the following are equivalent:

1. There exists a $d > 0$ such that $A_i^T d > 0$ for all $i = 1, 2, \ldots, m$.
2. There exists a $d > 0$ such that $C^T d > 0$ for all $C \in \mathcal{A}$.
3. Every matrix in $\mathcal{A}$ is positive stable (equivalently, a $\mathbf{P}$-matrix).
4. Every matrix in $\mathcal{A}^\#$ is positive stable (equivalently, a $\mathbf{P}$-matrix).
5. Every matrix in $\mathcal{A}$ has positive determinant.
6. Every matrix in $\mathcal{A}^\#$ has positive determinant.

Proof. If (1) holds, then for any matrix $C = \sum_{i=1}^m A_i D_i \in \mathcal{A}$, $C^T d = \sum_{i=1}^m D_i A_i^T d > 0$ by the imposed conditions on $D_i$. Thus condition (2) holds. Now each matrix $C$ in $\mathcal{A}$ is a $\mathbf{Z}$-matrix. The implication (2) ⇒ (3) follows from Remark 25.5.

The implications (3) ⇒ (4) ⇒ (5) are obvious.

The implication (5) ⇒ (6) follows from the identity (25.7). Finally, the implication (6) ⇒ (5) is obvious and the implication (5) ⇒ (1) follows from Lemma 25.15.

This completes the proof of the theorem. \qed

Several remarks are in order.
Remark 25.17. Thanks to Theorem 2, we can relate item (4) above to the uniqueness of solution in the vertical linear complementarity problem $\text{VLCP}(A^T, q)$ for any vector-tuple $q$. Here, $A^T = (A_1^T, A_2^T, \ldots, A_m^T)$. Item (5) is related to the so-called column $\mathcal{W}$-property and horizontal linear complementarity problems, see [29].

Remark 25.18. A conjecture due to Mason and Shorten [18] and D. Angeli says that for a set $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ of positive stable Z-matrices, the following are equivalent:

(a) Every matrix in the convex hull of $\mathcal{A}$ is positive stable.
(b) The switched system $\dot{x} + A_{\sigma}x = 0$ is (uniformly) asymptotically stable for arbitrary signals.

In [13], Gurvits, Shorten, and Mason disprove this conjecture by constructing a set $\{A_1, A_2\}$ of positive stable Z-matrices in $\mathbb{R}^{2 \times 2}$ for which item (a) holds but not (b). We note here that if we replace (a) by

(a') Every matrix in $\mathcal{A}^p$ is positive stable (or equivalently, every matrix in the convex set $\mathcal{A}$ is positive stable),

then (a') $\Rightarrow$ (b). This raises the question whether (b) $\Rightarrow$ (a'). Based on the work of Akar et al [1], we show that this is false.

Example 25.19. Consider the following positive stable Z-matrices

\[
A_1 = \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix}
1 & -\frac{1}{2} \\
-1 & 1
\end{bmatrix}.
\]

It is easily verified that every convex combination of $A_1$ and $A_2$ is positive stable. In this context (because every diagonal entry of $A_1$ and $A_2$ is one), by a result of Akar et al [1], the positive system $\dot{x} + A_{\sigma}x = 0$ is (uniformly) asymptotically stable for arbitrary signals. Thus condition (b) above holds. However, the column representative formed by the first column of $A_2$ and the second column of $A_1$ is not a P-matrix (that is, it is not positive stable). Hence (a') fails to hold.

Remark 25.20. It is well known that a Z-matrix $A$ is positive stable if and only if there is a diagonal matrix $D \succ 0$ such that $AD + DA^T \succ 0$. Now for a finite set $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ of Z-matrices, consider the following statements:

(i) Every row and column representative of $\mathcal{A}$ is positive stable.
(ii) There exists a diagonal matrix $D \succ 0$ such $A_i D + DA_i^T \succ 0$ for all $i = 1, 2, \ldots, m$.

We claim that (i) $\Rightarrow$ (ii) but not conversely. To see this, assume that (i) holds, in which case, by the above theorem, there exist vectors $u > 0$ and $v > 0$ in $\mathbb{R}^m$ such that

\[
A_i u > 0 \quad \text{and} \quad A_i^T v > 0 \quad (i = 1, 2, \ldots, m).
\]

Let $D$ be a diagonal matrix with $Dv = u$. Then $D$ has positive diagonal entries and

\[
(A_i D + DA_i^T) v = A_i u + D(A_i^T v) > 0
\]
for all $i$. This means that for each $i$, the symmetric $\mathbf{Z}$-matrix $A_iD + DA_i^T$ is a $\mathbf{P}$-matrix, hence $A_iD + DA_i^T \succ 0$. Thus item (ii) holds. To see that (ii) need not imply (i), we merely construct two $2 \times 2$ symmetric positive definite $\mathbf{Z}$-matrices $\{A_1, A_2\}$ such that a column representative of $\{A_1, A_2\}$ is not a $\mathbf{P}$-matrix. Then with $D = I$, item (ii) holds but not (i).

**Theorem 25.21.** Let $\mathcal{A}$ be a compact set of positive stable $\mathbf{Z}$-matrices in $\mathbb{R}^{n \times n}$. Then the following are equivalent:

1. There exists a $d > 0$ such that $A^T d > 0$ for all $A \in \mathcal{A}$.
2. Every column representative of $\mathcal{A}$ is a $\mathbf{P}$-matrix.
3. Every column representative of $\mathcal{A}$ has positive determinant.

**Proof.** The equivalence between (1) and (2) was proved by Song, Gowda, and Ravindran in [28]. The implication (2)⇒(3) is obvious and (3)⇒(2) follows from the previous theorem applied to a finite set of matrices.

**References**