Name:______________________

MATH152

Quiz 6 Solutions

date 07/06/2009

Total 80

Show all work legibly.

1. (20) Use a comparison or limit comparison to determine whether the series converges or diverges:

$$\sum_{k=2}^{\infty} \frac{k-1}{k^2+5}$$

Use limit comparison; compare to harmonic series.

$$\begin{aligned}
a_k &= \frac{k-1}{k^2+5} \\
b_k &= \frac{1}{k} \\
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{k-1}{k^2+5}\frac{k}{1} \\
&= \lim_{k\to\infty} \frac{k^2-k}{k^2+5} \\
&= 1
\end{aligned}$$

Since the limit is greater than 0, $\sum a_k$ and $\sum b_k$ either both converge or both diverge, by the Limit Comparison Test. Since the harmonic series diverges, so does $\sum_{k=2}^{\infty} \frac{k-1}{k^2+5}$

2. (20) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{k=1}^{\infty}(-1)^k \frac{5^k}{k!k^2}$$

Use the ratio test:

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\frac{5^{k+1}}{(k+1)!(k+1)^2}\frac{k!k^2}{5^k} \\
&= \lim_{k\to\infty}\frac{5k^2}{(k+1)^3} \\
&= 0
\end{aligned}$$

Since the limit is less than 1, the series is absolutely convergent, by the Ratio Test.

3. (20) Find the radius r and interval of convergence I of the following power series:

$$\sum_{k=1}^{\infty} \frac{1}{k}(x+2)^k$$

Apply the ratio test.

$$\begin{aligned} \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \frac{|x+2|^{k+1}}{k+1} \frac{k}{|x+2|^k} \\ &= |x+2| \lim_{k\to\infty} \frac{k}{k+1} \\ &= |x+2| \end{aligned}$$

By the Ratio Test, the series converges when $|x+2| < 1$. This occurs for, at least, all x in $(-3, 1)$. It remains to test the endpoints. For $x = -3$,

$$\sum_{k=1}^{\infty} \frac{1}{k}(-3+2)^k = \sum_{k=1}^{\infty} \frac{1}{k}(-1)^k.$$

This is the alternating harmonic series, which converges, by the Alternating Series Test. So $x = -3$ is included in the interval of convergence. For $x = -1$,

$$\sum_{k=1}^{\infty} \frac{1}{k}(-1+2)^k = \sum_{k=1}^{\infty} \frac{1}{k}.$$

This is the harmonic series, which is divergent. So $x = -1$ is not included in the interval of convergence.

$r = 2$

$I = [-3, -1)$

4. (20) Compute the Taylor series of $f(x) = e^{3x}$ about $x = 0$.

In the definition of the Taylor Series, let $c = 0$. The kth derivative of $f(x) = e^{3x}$ is $f^k(x) = 3^k e^{3x}$, so that $f^k(x) = 3^k$. Then the Taylor Series expansion is

$$e^{3x} = \sum_{k=0}^{\infty} \frac{3^k}{k!} x^k$$