Lectures 11 through 14 – contents

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- Motivation: exact methods for solving a large system of linear equations have complexity $O(n^3)$ and memory requirements $O(n^2)$, both of which may be impossible to handle by available machines.

- Basic idea: if $A = I - B$ with $\|B\| < 1$, then the solution of $Ax = b$ is

$$x = (I - B)^{-1}b = \sum_{k=0}^{\infty} B^k b.$$  

For practical purposes it may be sufficient to compute $x_n = \sum_{k=0}^{n} B^k b$. Note that

$$x_{n+1} = b + Bx_n = x_n + (b - Ax_n).$$

Thus we arrive at the idea of the basic iteration:

$$x_{n+1} = x_n + Q^{-1}(b - Ax_n),$$

where $Q \approx A$ is a matrix that is easy to invert.

- Error propagation: if $e_n = x_n - x^*$ is the error, with $x^* = A^{-1}b$ being the exact solution, then

$$e_{n+1} = (I - Q^{-1}A)e_n = \ldots (I - Q^{-1}A)^{n+1}e_0.$$  

Therefore, if $(I - Q^{-1}A)^n \to 0$ (as a sequence of matrices), then $e_n \to 0$, and $x_n \to x^*$. This is the case when

$$\|I - Q^{-1}A\| < 1,$$

for some matrix norm $\| \cdot \|$. 


If $A$ is diagonally dominant, and $Q = D = \text{diag}(A)$, then 
\[ |I - Q^{-1}A|_{\infty} < 1, \]

hence the corresponding simple iteration converges. This choice leads to the Jacobi iteration.

**Lecture 12. Iterative methods for linear systems. The Gauss-Seidel method.**

- Spectrum and spectral radius of a matrix $(\sigma(A), \rho(A))$.
- **Theorem:**

  \[ \rho(A) = \inf \{ \|A\| : \|\cdot\| \text{ is a matrix norm} \} . \]

  As a consequence, if $\rho(I - Q^{-1}A) < 1$, then $(I - Q^{-1}A)^n \to 0$.

- The choice $Q = L$ (the lower triangular part of $A$ including the diagonal) leads to the Gauss-Seidel method. We proved that if $A$ is diagonally dominant, then the Gauss-Seidel method converges, by showing that $\rho(I - L^{-1}A) < 1$.

- Case study: consider the two-point boundary value problem:

  \[
  \begin{cases}
  -\partial_{xx}^2 u = f \\
  u(0) = u(1) = 0
  \end{cases}
  \]  (3)

  We discretize the problem (3) by representing the unknown $u$ by its values at the gridpoints $x_i = i/n$, $0 \leq i \leq n$, and by using the approximation

  \[ \partial_{xx}^2 u = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}, \]

  where $h = 1/n$. The resulting linear system is $AU = B$ where

  \[
  A = \begin{bmatrix}
  2 & -1 & 0 & \ldots & 0 & 0 \\
  -1 & 2 & -1 & \ldots & 0 & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & 0 & \ldots & -1 & 2
  \end{bmatrix}
  \]  (4)

  is an $(n-1) \times (n-1)$ matrix, and $B_i = h^2 f(i/n)$. $A$ is not diagonally dominant, but we have verified numerically that both the Jacobi and Gauss-Seidel iterations converge, but slowly. We have also seen that the convergence quality decreases with increasing $n$. 

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- $A$ from (4) is symmetric positive definite (SPD).

- Basics of symmetric matrices (diagonalizable: real eigenvalues, orthonormal basis of eigenvectors). It follows that $\rho(A) = \|A\|_2$.

- **Theorem:** if $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of a symmetric matrix $A$, and $0 < \alpha < 2/\lambda_n$, then the iteration

$$x_{n+1} = x_n + \alpha (b - Ax_n) \quad (5)$$

converges to the solution $x^* = A^{-1}b$. The optimal choice for $\alpha$ is

$$\alpha = \frac{2}{\lambda_1 + \lambda_n},$$

case in which the convergence rate in the $\| \cdot \|_2$ is

$$\rho(I - \alpha A) = \frac{\lambda_n - 1}{\lambda_n + 1} = \frac{\kappa(A) - 1}{\kappa(A) + 1}.$$

- Condition number of a matrix:

$$\kappa(A) = \|A^{-1}\| \cdot \|A\|.$$


- The spectrum of $A$ from (4) is in $[Ch^2, 4]$, therefore Jacobi uses almost the optimal $\alpha$; $\text{cond}(A) = Ch^{-2}$.

- The conjugate gradient algorithm.