Lectures 16 – contents

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Lecture 16. (Section 6.1)

• **The problem:** Given some information about a function \( f \) – values, and/or derivatives at a set of points \( x_0, x_1, x_2, \ldots, x_n \) – we would like additional information at, maybe different points, e.g., \( f(x), f''(x_2) \).

• **The solution:** Represent the function by a function \( P \) for which we have an analytical expression, a process called interpolation.

• **Polynomial interpolation:** \( P \) is globally polynomial.

• **Theorem:** Given \( x_0, x_1, x_2, \ldots, x_n \) pairwise distinct numbers on the real line, and values \( y_0, y_1, \ldots, y_n \), there exists a unique polynomial of degree \( \leq n \) such that \( P(x_i) = y_i, \forall 0 \leq i \leq n \). (Proof is done by the “counting” argument and the fact that it’s a linear system that needs to be solved.) \( P \) is called the interpolation polynomial.

• Newton form of the interpolation polynomial: we construct a sequence of polynomials \( p_0, p_1, \ldots, p_n \), with \( P = p_n \), and

\[
p_k(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) + \cdots + c_k(x-x_0) \cdots (x-x_k-1).
\]

Computing \( c_i \)'s is done recursively:

\[
c_k = \frac{y_k - p_{k-1}(x_k)}{(x_k - x_0) \cdots (x_k - x_{k-1})}.
\]

• Computing \( p_k(x) \) through nested multiplication:

\[
p_k(x) = \ldots (x-x_{k-1})c_k + c_{k-1}(x-x_{k-2}) + c_{k-3}(x-x_{k-3}) + \cdots + c_0.
\]
• **Lagrange formula:**

\[ P(x) = \sum_{i=0}^{n} y_i l_k(x), \]

where the *cardinal functions* are

\[ l_k(x) = \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}. \]

• **Example:** \( f(x) = |x|, x_0 = -1, x_1 = 0, x_2 = 1, y_i = f(x_i). \)

• **Theorem on polynomial interpolation error:** If \( f \) is \( n + 1 \) times differentiable in \([a, b]\), \( x_0, \ldots, x_n \in [a, b] \), then for each \( x \in [a, b] \) there exists \( \xi_x \in (a, b) \) s.t.

\[ f(x) - P(x) = \frac{1}{(n + 1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i). \]