1. (25 pts) Given the matrix \( A = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} \)

   a) (5 pts) Compute the characteristic polynomial of \( A \).

   **Answer:**
   \[
   \text{char}(A) = \det(A - \lambda I) = \det \begin{pmatrix} 2 - \lambda & 2 \\ -2 & 2 - \lambda \end{pmatrix} = (2-\lambda)(2-\lambda) - (2)(-2) = \lambda^2 - 4\lambda + 8
   \]
   (Check: Note that \( \text{char}(0) = 0^2 - 4(0) + 8 = \det(A) \))

   b) (5 pts) Find the eigenvalues of \( A \).

   **Answer:**
   Find the roots (zeros) of the char poly, \( \text{char}(A) = \lambda^2 - 4\lambda + 8 \).

   As this is quadratic (as \( A \) was 2x2) we may use the quadratic formula:
   \[
   \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)} = 2 \pm \frac{\sqrt{16}}{2} = \{2 + 2i, 2 - 2i\}
   \]
   (Check: \( \det(A) = \text{product of eigenvalues: } \lambda_1\lambda_2 = (2+2i)(2-2i) = 4-4i^2 = 8 \).)

   c) (10 pts) Find the eigenvectors of \( A \).

   **Answer:**
   For each eigenvalue \( \lambda \), find a basis for the null space of \( (A - \lambda I) \)

   \( \lambda = 2+2i: \)

   \[
   (A - \lambda I) = (A - (2+2i)I) = \begin{pmatrix} 2 - (2 + 2i) & 2 \\ -2 & 2 - (2 + 2i) \end{pmatrix} = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}
   \]

   The (1,1) element is the only pivot so, while the first column is a pivot column the second column is a non-pivot column and \( y \) is a free variable.

   From the first row we get the equation \((-2i)x + 2y = 0 \) so \( x = -2y/(-2i) = iy \).

   An element of the null space is: \( \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} iy \\ y \end{pmatrix} = y \begin{pmatrix} -i \\ 1 \end{pmatrix} \), so \( \begin{pmatrix} -i \\ 1 \end{pmatrix} \) is a reasonable basis (as is any other multiple such as \( \begin{pmatrix} -2i \\ 2 \end{pmatrix} \) or \( \begin{pmatrix} 1 \\ i \end{pmatrix} \)).

   (Check: \( \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -i \\ 1 \end{pmatrix} = (2+2i)\begin{pmatrix} -i \\ 1 \end{pmatrix} \).

   \( \lambda = 2-2i: \)
(Quick Soln: Given a complex conjugate pair of eigenvalues \( \lambda_1 \) and \( \lambda_2 = \lambda_1^* \), they have complex conjugate eigenvectors. Thus, as \( \begin{pmatrix} -i \\ 1 \end{pmatrix} \) is an eigenvector of the eigenvalue \((2+2i)\), the vector \( \begin{pmatrix} -i \\ 1 \end{pmatrix} \) is an eigenvector of \((2-2i)\).)

\[
(A - \lambda I) = (A - (2 - 2i)I) = \begin{pmatrix} 2 & 2 \\ -2 & 2 - (2 - 2i) \end{pmatrix} = \begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix}
\]

The \((1,1)\) element is the only pivot so, while the first column is a pivot column the second column is a non-pivot column and \( y \) is a free variable. From the first row we get the equation \((2i)x + 2y = 0\) so \( x = 2y/(-2i) = iy \).

An element of the null space is:
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} iy \\ 1 \end{pmatrix}, \quad \text{so} \quad \begin{pmatrix} i \\ 1 \end{pmatrix}
\]

is a reasonable basis (as is any other multiple such as \( \begin{pmatrix} 3i \\ 3 \end{pmatrix} \) or \( \begin{pmatrix} -1 \\ i \end{pmatrix} \)).

(Check: \( \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = (2 - 2i) \begin{pmatrix} i \\ 1 \end{pmatrix} \).)

\[d) \quad (5 \text{ pts}) \text{ Find a matrix } P \text{ such that } P^{-1}AP \text{ is diagonal and compute } P^{-1}AP.\]

\textbf{Answer:} \ We build \( P \) with the eigenvectors as columns and then compute \( P^{-1} \):

\[
P = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}
\]

and we compute \( P^{-1} \) by row reduction:

\[
\begin{pmatrix} -i & i & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -i & i & 1 & 0 \\ 0 & 2 & -i & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -i & 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 2 & -i & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & \frac{i}{2} & \frac{1}{2} \\ 0 & 1 & \frac{i}{2} & \frac{1}{2} \end{pmatrix}
\]

So \( P^{-1} = \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} \) and \( P^{-1}AP = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -i & i \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 2 + 2i & 0 \\ 0 & 2 - 2i \end{pmatrix} \)

(Check: The diagonal elements are the eigenvalues.)

\[2. \quad (25 \text{ pts}) \text{ Given the matrix } A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ -1 & -2 & 5 \end{pmatrix}\]

\textbf{Answer:} \ Before going any further we row reduce the matrix \( A \).
We note that there are pivots in the first and third columns.

a) (6 pts) Compute \( \text{rank}(A) \).

Answer: The rank, defined as the dimension of the column space of \( A \), can be computed as the number of pivot elements. Thus \( \text{rank}(A) = 2 \).

b) (6 pts) Find a basis of the column space of \( A \).

Answer: The pivot columns of the original matrix \( A \) form a basis of the column space (Thm 6, Sect 4.3, Lay). Thus the basis of \( \text{col}(A) \) is the first and third columns,
\[
\begin{pmatrix}
1 \\ 2 \\ -1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
3 \\ 2 \\ 5
\end{pmatrix}.
\]

3. (25 pts) Prove that if \( A \) has an eigenvalue \( \lambda \) then the matrix \( A^3+2A^2+A \) has an eigenvalue \( \lambda^3+2 \lambda^2+\lambda \).

Proof: Let \( x \) be the eigenvector of \( A \) corresponding to the eigenvalue \( \lambda \), so \( Ax = \lambda x \).
Then \((A^3+2A^2+A)x = A^3x+2A^2x+Ax = A^2(Ax)+2A(Ax)+\lambda x = \lambda A(Ax)+2\lambda(Ax)+\lambda x = \lambda^2(Ax)+2\lambda^2x+\lambda x = \lambda^3x+2\lambda^2x+\lambda x = (\lambda^3+2\lambda^2+\lambda)x\)

So \((A^3+2A^2+A)x = (\lambda^3+2\lambda^2+\lambda)x\) and the matrix \(A^3+2A^2+A\) has an eigenvalue \(\lambda^3+2\lambda^2+\lambda\) with eigenvector \(x\).

4. (25 pts) Multiple Choice
   a) The rank of a 3x3 non-zero matrix is greater than zero.
      ALWAYS/NEVER/SOMETIMES
      Answer: ALWAYS – If the matrix is non-zero at least one column is non-zero.
      Thus \(\text{rank} = \dim(\text{col}(A)) > 0\).

   b) If \(B=\{b_1,b_2,b_3\}\) is a basis of \(\mathbb{R}^3\) then \(\{Ab_1,Ab_2,Ab_3\}\) is a basis of \(\mathbb{R}^3\).
      ALWAYS/NEVER/SOMETIMES
      Answer: SOMETIMES – If \(A\) is invertible this is true, an obvious example being the case where \(A\) is the identity matrix. If \(A\) is non-invertible this is false, an obvious example being the case where \(A\) is the zero matrix.

   c) A change of coordinates matrix is invertible: ALWAYS/NEVER/SOMETIMES
      Answer: ALWAYS – Recall that the columns of a change of coords matrix are the coordinates of the vectors of one basis when expressed in terms of another basis. We may always construct such a matrix (including the inverse) because the vectors of a basis span the space (including the vectors of the other basis).

   d) Similar matrices have the same eigenvectors. ALWAYS/NEVER/SOMETIMES
      Answer: SOMETIMES – While similar matrices have the same eigenvalues they do not necessarily have the same eigenvectors.

   e) Given that the matrix \(A\) is not diagonalizable. The product of the eigenvalues of \(A\) is equal to the determinant of \(A\). ALWAYS/NEVER/SOMETIMES
      Answer: ALWAYS – The product of the eigenvalues of a matrix is equal to the determinant for all matrices, not just diagonalizable matrices. For non-diagonalizable matrices this can be seen by reducing the matrix to Jordan Form and observing that this is true.