1. (25 pts) Let $A = \begin{pmatrix} 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

   a) Compute $\det(A)$ by Gaussian Elimination.

   $A_1 = \begin{pmatrix} 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

   $\text{swap}(3,2) \rightarrow A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

   $\text{swap}(2,4) \rightarrow A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

   $\text{(iii)} \rightarrow \frac{1}{2} (\text{iv}) \rightarrow A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

   As $A_4$ is triangular its determinant is the product of the diagonal elements, so $\det(A_4) = (1)(1)(2)(2) = 4$.

   The transformation which takes $A_3$ to $A_4$ is a replacement, so $\det(A_3) = \det(A_4) = 4$.

   The transformation which takes $A_2$ to $A_3$ is a swap, so $\det(A_2) = -\det(A_3) = -4$.

   The transformation which takes $A_2$ to $A_3$ is a swap, so $\det(A_1) = -\det(A_2) = 4$.

   So the determinant of $A$ is 4.

   b) Compute $\det(A)$ by Cofactor expansion.

   We note that the first column of $A$ is low density, so we expand on that column.

   $\det(A) = \sum a_{i,1}(-1)^{i+1} A_{i,1} = 0 + a_{2,1}(-1)^{2+1} A_{2,1} + 0 + 0$

   $= (1)(-1)^3\begin{vmatrix} 0 & 1 & 3 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -4$

   We can compute the determinant of this minor by cofactor expansion, noting that the first column is again of low density. Calling this matrix $B$ for convenience we get:

   $\det(B) = \begin{vmatrix} 0 & 1 & 3 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = ((1)(2) - (3)(2)) = -4$

   If $\det(B) = -4$, then $\det(A) = 4$. 

Check: Compare the results of the two methods of computing \( \det(A) \).

2. (25 pts) If \( B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 4 & 3 \end{pmatrix} \)

a) Compute an LU decomposition (without pivoting) of \( B \)

First we compute both \( U \) by applying a sequence of elementary row operations to \( B \):

\[
\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 1 & 4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 2 & 2 \end{pmatrix} = B_2
\]

\[
\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{pmatrix} = U
\]

There are several ways to think about how we compute \( L \):

- Using the computational approach of Lay seen in Example 2, page 136, we note that the first column of \( L \) is the first column of \( B \) (below the diagonal) divided by the pivot element, i.e. \( (1,2,1)/1 = (1,2,1) \). We compute the second column of \( B \) by looking at \( B_2 \), the place where we start clearing out the second column, and making the second column of \( L \) the second column of \( B_2 \) below the diagonal, divided by the second pivot, or \( (0,-2,2)/(-2) = (0,1,-1) \). The third column is easily computed as the only element below the diagonal in that column is the pivot itself. Thus the third column of \( L \) is \( (0,0,1)/1 = (0,0,1) \). Thus

\[
L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}
\]

- Alternatively, we may note that as we compute \( U = E_3 E_2 E_1 B \) and require that \( B = LU = L(E_3 E_2 E_1 B) \), we see that \( I = L E_3 E_2 E_1 B \), or \( L = E_1^{-1} E_2^{-1} E_3^{-1} \).

Here \( E_i \) is the replacement of row \( (ii) \) by row \( (ii) \) minus 2 times row \( (i) \) minus 2 times row \( (i) \), and so on. Thus we compute \( L \) by applying the reverse sequence of inverse elementary operations to the identity matrix \( I \):

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}
\]

Check: Check by multiplying \( LU \) and confirming that \( LU = B \).

b) Solve \( Bx = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} \) for the vector \( x \) using the LU decomposition of \( B \).
We solve the equation \( Bx = (LU)x = b \) in two steps, first letting \( y = Ux \) and solving \( Ly = b \) to recover \( y \), and then solving \( Ux = y \) to recover the value of \( x \).

\[
Ly = \begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
1 & -1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
\end{pmatrix} = \begin{pmatrix}
1 \\
-1 \\
5 \\
\end{pmatrix} = b
\]

We solve this by back-solving, starting with the equation from the first row that:

\[y_1 = 1\]

The second row gives us the equation:

\[2y_1 + y_2 = -1 \text{ or } y_2 = -1 - 2y_1 = -1 - 2(1) = -3\]

The third row gives us the equation:

\[y_1 + (-1)y_2 + y_3 = 5 \text{ or } y_3 = 5 - y_1 - (-1)y_2 = 5 - (1) - (-1)(-3) = 1\]

Thus \( y = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \), which can be checked by multiplying and confirming that \( Ly = b \).

\[Ux = \begin{pmatrix}
1 & 2 & 1 \\
0 & -2 & -1 \\
0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix} = \begin{pmatrix}
1 \\
-3 \\
1 \\
\end{pmatrix} = y
\]

We solve this by back-solving, starting with the equation from the last row:

\[x_3 = 1\]

The second row gives us the equation:

\[(-2)x_2 + (-1)x_3 = -3 \text{ or } (-2)x_2 = -3 - (-1)x_3\]

so \( x_2 = (-3 - (-1)x_3)/(-2) = (-3 - (-1)(1))/(-2) = 1 \)

The first row gives us the equation:

\[x_1 + (2)x_2 + x_3 = 1 \text{ or } x_1 = 1 - (2)x_2 - x_3 = 1 - (2)(1) - (1) = -2\]

Thus \( x = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \), which can be checked by multiplying and confirming that \( Ux = y \).

**Check:** We finally check by multiplying and confirming that \( Bx = b \)

3. (25 pts) Compute the inverse of \( A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \) by Gaussian Elimination.

We compute the inverse by bringing the augmented matrix \((A | I)\) to reduced row echelon form:

\[
(A | I) = \begin{pmatrix}
1 & 2 & 1 & 1 & 0 & 0 \\
2 & 2 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\rightarrow \begin{pmatrix}
1 & 2 & 1 & 1 & 0 & 0 \\
0 & -2 & -1 & -2 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\]
Thus \( A^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} \)

Check: Multiply and confirm that \( AA^{-1} = I \).
Note that at any step of the computation \((C \mid D)\) we may confirm the computation to that point by confirming that \(DA = C\).

4. (25 pts) Multiple Choice (5 pts/correct answer, -2 pts/incorrect answer)

a) If \( E \) is an elementary matrix then \( \text{det}(E) = 1 \): ALWAYS/NEVER/SOMETIMES

**Answer:** SOMETIMES - Of the three types of elementary matrices, a swap matrix has \( \text{det}(E) = -1 \), a replacement matrix has \( \text{det}(E) = 1 \) and a scale by \( \lambda \) has a determinant of \( \lambda \).

b) If \( A \) is invertible and the constant \( c \) is not 0 then \( cA \) is invertible: ALWAYS/NEVER/SOMETIMES

**Answer:** ALWAYS - If \( A \) is invertible it has an inverse \( A^{-1} \). The inverse of \( (cA) \) is the matrix \( (c^{-1}A^{-1}) \).

c) If \( AB = I \) then \( BA = I \): ALWAYS/NEVER/SOMETIMES

**Answer:** SOMETIMES - This is true if \( A \) and \( B \) are square but not necessarily true otherwise. Consider the case (partly from homework) of the matrices:

\[
A = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}
\]

where we see that
\[ AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ but } BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} \]

d) If the linear transformation \( T \) is invertible then \( T \) is onto.

**Answer:** ALWAYS - If \( T \) is invertible there must be some linear transformation \( S \) such that \( S(T(x)) = x \) and \( T(S(y)) = y \). Recall that \( T \) is onto if for any \( y \) there must be some \( x \) such that \( T(x) = y \). Given any \( y \) the value \( x = S(y) \) will satisfy this requirement. Thus \( T \) must be onto.

e) If \( \det(A) = d \) then \( \det(-A) = d \). ALWAYS/NEVER/SOMETIMES

**Answer:** SOMETIMES - If the dimensions of \( A \) are even then \( \det(-A) = \det(A) \). If the dimensions of \( A \) are odd then \( \det(-A) = -\det(A) \). We see this by converting \( A \) to \(-A\) by scaling one row at a time by multiplying by a scaling factor of \((-1)\). At each step this multiplies the determinant by a factor of \((-1)\). Thus the determinant is unchanged if there are an even number of steps, in other words if \( A \) has an even dimension.