1. (25 pts)

a) Compute \( \det \begin{pmatrix} 5 & 5 & 10 \\ 1 & 1 & 3 \\ 1 & 3 & 4 \end{pmatrix} \) by Gaussian Elimination.

**Answer:** We perform Gaussian reduction, noting at each step what effect each elementary row operation has on the determinant:

\[
\begin{align*}
\det \begin{pmatrix} 5 & 5 & 10 \\ 1 & 1 & 3 \\ 1 & 3 & 4 \end{pmatrix} & \xrightarrow{(x,1)} \det \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \\ 1 & 3 & 4 \end{pmatrix} \\
& \xrightarrow{(x,-1)(i)} \det \begin{pmatrix} 5 & 5 & 10 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \\
& \xrightarrow{(x,1)(ii)} \det \begin{pmatrix} 5 & 5 & 10 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \\
\end{align*}
\]

As this last matrix is triangular its determinant is the product of the diagonal elements, \( \det = (5)(2)(1) = 10 \). In order to compute the determinant of the original matrix we back up each step's effect on the determinant:

\[
\det = 10/1/(-1)/(i) = -10.
\]

(We can check this result by using the 3x3 “shortcut” method: \( \det = (5)(1)(4)+(5)(3)(1)+(1)(0)(10)-(10)(1)(3)-(10)(1)(1)-(5)(1)(4)-(5)(3)(3) = -10 \).)

b) Compute \( \det \begin{pmatrix} 5 & 5 & 10 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \) by Cofactor expansion.

**Answer:** We find the sparsest row or column to expand over. Either the third row or the second column would be appropriate. (We could expand over any row or column, but this choice will minimize our work.) We expand along the third row as follows:

\[
\det \begin{pmatrix} 5 & 5 & 10 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} = (1)(-1)^{3+1} \det \begin{pmatrix} 5 & 10 \\ 1 & 3 \end{pmatrix} + (0)(-1)^{3+2} \det \begin{pmatrix} 5 & 10 \\ 1 & 3 \end{pmatrix} + (2)(-1)^{3+3} \det \begin{pmatrix} 5 & 5 \\ 1 & 1 \end{pmatrix}
\]

At this point we could expand the 2x2 determinants any way, but we choose to stick to the method of cofactors, expanding each 2x2 determinant over the first row:

\[
\begin{align*}
= & (1)(-1)^{3+1}[(5)(-1)^{1+1}(3) + (10)(-1)^{1+2}(1)] + (0)(-1)^{3+2}[(5)(-1)^{1+1}(3) + (10)(-1)^{1+2}(1)] + (2)(-1)^{3+3}[(5)(-1)^{1+1}(1) + (10)(-1)^{1+2}(1)] \\
= & 5
\end{align*}
\]

(Again, as this determinant is only 3x3 we may check the result: \( \det = (5)(1)(2) + (5)(3)(1) + (1)(0)(10) - (10)(1)(1) - (5)(1)(2) - (3)(0)(5) = 5 \).)
2. (25 pts) If $B = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 4 & 3 \end{pmatrix}$

- Perform an LU decomposition (without pivoting) on $B$

**Answer:** We do this in both of two ways:
First the book’s method, in which we generate a column of $L$ at each step by dividing the column of $U$ by the pivot element:
\[
\begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}
\]
where the first column of $L$ (below the diagonal) is the first column of the original $B$, (1,3,1), divided by the pivot, 1.
\[
\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}
\]
where the second column of $L$ (below the diagonal) is the second column of the preceding state of $U$ below the diagonal, (-4,2), divided by the pivot, -4.

A second method (more appropriate for teaching than computation) is to perform the Gaussian reduction with no scaling or swaps then solve for the values of $L$:
\[
\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} x & 1 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}
\]
where $x$, $y$ and $z$ are the unknown values in $L$. By solving for them in the appropriate order we may do it with little effort:
The product of the second row of $L$ and the first column of $U$ gives the (2,1) element of $B$:
\[
(x)(1)+(1)(0)+(0)(0)=3, \text{ so } x=3
\]
The product of the third row of $L$ and the first column of $U$ gives the (3,1) element of $B$:
\[
(y)(1)+(z)(0)+(0)(0)=1, \text{ so } y=1
\]
The product of the third row of $L$ and the second column of $U$ gives the (3,2) element of $B$:
\[
(y)(2)+(z)(-4)+(0)(0)=4, \text{ so } (1)(2)+(z)(-4)+(1)(0)=4, \text{ so } z=-1/2
\]
And we recover the same matrices $L$ and $U$ as in the first method.
(These computations may be checked by confirming that $LU=B$.)

- Solve $Bx = \begin{pmatrix} 8 \\ 10 \\ 18 \end{pmatrix}$ for the vector $x$ using the LU decomposition computed.

**Answer:** From our earlier work we have $B=LU$ so $Bx=LUx$. We define $y=Ux$ as usual, so we now need to solve the equation $Ly=(8,10,18)$. As $L$ is lower triangular this is best solved from the top down as follows:
From the first row of $L$: $(1)y_1 + (0)y_2 + (0)y_3 = 8$, so $y_1 = 8$
From the second row: $(3)y_1 + (1)y_2 + (0)y_3 = 10$, so $(3)(8) + (1)y_2 + (0)y_3 = 10$
and $y_2 = -14$
From the third row: \( (1)y_1 + (-1/2)y_2 + (1)y_3 = 18 \), so \( (1)(8) + (-1/2)(-14) + (1)y_3 = 18 \) and \( y_2 = 3 \)

Thus \( Ux = y = \begin{pmatrix} 8 \\ -14 \\ 3 \end{pmatrix} \)

Now we solve this equation for \( x \). As \( U \) is upper triangular we can most easily solve this in sequence from the bottom up:

From the third row of \( U \): \( (0)x_1 + (0)x_2 + (1)x_3 = 3 \), so \( x_3 = 3 \)

From the second row: \( (0)x_1 + (-4)x_2 + (-2)x_3 = -14 \), so \( (0)x_1 + (-4)x_2 + (-2)(3) = -14 \) and \( x_2 = 2 \)

From the first row: \( (1)x_1 + (2)x_2 + (1)x_3 = 8 \), so \( (1)x_1 + (2)(2) + (1)(3) = 8 \) and \( x_1 = 1 \)

Thus our solution is \( x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \), which can be checked by confirming that \( Bx = \begin{pmatrix} 8 \\ 10 \\ 18 \end{pmatrix} \)

3. (25 pts) Compute the inverse of \( A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix} \).

**Answer:** We compute the inverse by reducing \( A \) to reduced echelon form, keeping track of how the same elementary row operations act on the identity matrix:

\[
\begin{pmatrix} 1 & -2 & 1 & | & 1 & 0 & 0 \\ 2 & 2 & 1 & | & 0 & 1 & 0 \\ 1 & 2 & 2 & | & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 & | & 1 & 0 & 0 \\ 0 & 6 & -1 & | & -2 & 1 & 0 \\ 0 & 0 & 5 & | & 1 & 5 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1/5 & 3/5 & -2/5 \\ 0 & 6 & 0 & | & 4/5 & 2/5 & 3/5 \\ 0 & 0 & 5 & | & 1/3 & 1/3 & -2/3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1/5 & 3/5 & -2/5 \\ 0 & 1 & 0 & | & -3/10 & 1/10 & 1/5 \\ 0 & 0 & 1 & | & 1/3 & -2/3 & 3/5 \end{pmatrix}
\]

Thus \( A^{-1} = \begin{pmatrix} 1/5 & 3/5 & -2/5 \\ -3/10 & 1/10 & 1/5 \\ 1/3 & -2/3 & 3/5 \end{pmatrix} \)

(Work is checked by confirming that \( AA^{-1} = I \))

4. (25 pts) Multiple Choice

a) If \( AB = AC \) where \( A \), \( B \) and \( C \) are all \( nxn \) matrices and \( A \) is not the zero matrix then \( B = C. \) **ALWAYS/NEVER/SOMETIMES**

**Answer:** **SOMETIMES** – If \( A \) is invertible then \( B = C. \) There are cases where \( B \) and \( C \) are not equal, e.g.: \( A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( C = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \)
b) If the matrix $A$ is invertible then $Ax=0$ has non-trivial solutions. 
**Answer:** NEVER (see Thm 8, sect 2.3)

c) If $A$ is an elementary matrix then $\det(A)=1$:  ALWAYS/NEVER/SOMETIMES 
**Answer:** SOMETIMES – We consider the three types of elementary matrices:  
- row swap: $\det = -1$  
- row scale: $\det = \lambda$  
- subtract multiple of row: $\det = 1$

d) If the columns of $A$ span the co-domain of $A$ then $A$ is onto.  
**Answer:** ALWAYS – The span of the columns of $A$ is the range of $A$. If the range is the same as the co-domain then $A$ is onto.

e) If $AB=C$, where $A$, $B$ and $C$ are $n \times n$ matrices, where $C$ is invertible then $A$ is invertible.  ALWAYS/NEVER/SOMETIMES 
**Answer:** ALWAYS – We recall that if $AB=C$ then $\det(A)\det(B)=\det(C)$. If $C$ is invertible then $\det(C)$ is not zero. Thus $\det(A)$ is also not zero, but this means that $A$ is invertible.