1. (25 pts) Is \( \mathbf{u} = \begin{pmatrix} -3 \\ 5 \\ 3 \end{pmatrix} \) in the span of \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \), \( \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \), \( \mathbf{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \)?

If so, for what weights \( \{w_1,w_2,w_3\} \) does \( \mathbf{u} = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + w_3 \mathbf{v}_3 \)?

**Answer:** We solve the second question in order to answer the first question. Set up the problem \( \mathbf{u} = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + w_3 \mathbf{v}_3 \) as the linear system

\[
\begin{bmatrix}
1 & 2 & -1 \\
2 & 1 & 2 \\
3 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3
\end{bmatrix}
= \begin{bmatrix}
-3 \\
-5 \\
3
\end{bmatrix},
\]

or equivalently the augmented matrix

\[
\begin{bmatrix}
1 & 2 & -1 & -3 \\
2 & 1 & 2 & 5 \\
3 & 2 & 1 & 3
\end{bmatrix}
\]

This is now brought to reduced echelon form as:

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

We see that this is a consistent system as there is no row of form \((0,0,0|1)\). We can also read off the solution to the second question: \( \{w_1,w_2,w_3\} = \{1,-1,2\} \) and \( \mathbf{u} = (1)\mathbf{v}_1 + (-1)\mathbf{v}_2 + (3)\mathbf{v}_3 \).

2. (25 pts) If \( \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \)

a) Bring this matrix to reduced echelon form
b) Which columns have pivots?
c) Does the equation \( \mathbf{A}\mathbf{x}=\mathbf{0} \) have non-trivial solutions?
d) If \( \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \) and \( \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), find all solutions to \( \mathbf{A}\mathbf{x}=\mathbf{b} \).
e) In (d) which variables are free?

**Answer:** Gaussian reduction gets us reduced echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 & -3/2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1/2
\end{bmatrix}
\]

We see that the first, second and third columns have pivots, but the fourth does not.
As the fourth column does not have a pivot the fourth variable is free and the equation $Ax=0$ has non-trivial solutions of the form $\tilde{x} = t \begin{pmatrix} 3/2 \\ -2 \\ -1/2 \\ 1 \end{pmatrix}$, where $t$ is an arbitrary parameter. Now, coming to part (d) we see that we probably should have done Gaussian reduction on an augmented matrix to begin with. We go back and solve the system

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 1 \\
2 & 3 & 2 & 4 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

reducing it to the form

$$
\begin{pmatrix}
1 & 0 & 0 & -3/2 & 3/2 \\
0 & 1 & 0 & 2 & -1 \\
0 & 0 & 1 & 1/2 & 1/2
\end{pmatrix}
$$

From this we may easily read off a solution $x_1=3/2$, $x_2=-1$, $x_3=1/2$, $x_4=0$. Adding this (or any other) particular solution to the homogeneous solution we get a general solution of $\tilde{x} = t \begin{pmatrix} 3/2 \\ -1 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3/2 \\ -2 \\ -1/2 \\ 1 \end{pmatrix}$.

Note here that there are many ways to write this general solution – either the choice of particular solution may vary or we may replace $t$ by any multiple of $t$. For example, two other ways of writing the general solution are:

$$
\tilde{x} = t \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 3/2 \\ -2 \\ -1/2 \\ 1 \end{pmatrix} \quad \text{and} \quad \tilde{x} = t \begin{pmatrix} 3/2 \\ -1 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \\ -1 \\ 2 \end{pmatrix}.
$$

The free variable in (d) was $x_4$, which we got when we saw that the fourth column had no pivot. (While we wrote the solution in terms of an indeterminate $t$, we could have just as easily replaced $t$ with $x_4$.)

3. (25 pts) If $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

a) Find all solutions to $Ax=0$.
b) Find all solutions to $Ax=\vec{b}$.

**Answer:** Rather than solving the homogeneous system first and then have to repeat our work for the inhomogeneous system we start by performing Gaussian reduction on the augmented matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 3 & 2 \\
1 & -1 & 3 & 1
\end{pmatrix}
$$

arriving at the reduced echelon matrix
Looking at the non-augmented part of the matrix (the left-hand three columns) we see that the third column is non-pivot, so \( x_3 \) is a free variable.

Thus the solution to the homogeneous system (part (a)) is \( \bar{x} = t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \), where \( t \) is the usual parametrization variable.

From the reduced echelon matrix we may easily read off a particular solution \( x_1 = 1, \ x_2 = 0, \ x_3 = 0 \). Adding this particular solution (or any other) to the homogeneous solution we get the general solution to the inhomogeneous equation (part (b))

\[
\bar{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.
\]

4. (25 pts) Multiple Choice

a) The system of equations \[
\begin{pmatrix}
a & b & c \\
d & e & f
\end{pmatrix}
\begin{pmatrix}
x \\ y
\end{pmatrix}
=
\begin{pmatrix}
1 \\ 2
\end{pmatrix}
\]
is consistent: YES/NO/SOMETIMES

**Answer:** SOMETIMES - While the shape of the system (#columns>#rows or #vars>#constraints) suggests that in most cases it will be consistent, this is not always the case, in particular where the rows are linear multiples. This would occur if for some parameter \( t \) not equal to 2 we have \( d=at, \ e=bt \) and \( f=ct \). For example, the system

\[
\begin{pmatrix}
1 & 2 & 3 \\
3 & 6 & 9
\end{pmatrix}
\begin{pmatrix}
x \\ y
\end{pmatrix}
=
\begin{pmatrix}
1 \\ 2
\end{pmatrix}
\]

is not consistent (reduce it and see). A more obvious case is \( d=e=f=0 \), which is not consistent.

b) The linear map \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is onto: YES/NO/SOMETIMES

**Answer:** NO – Such a map is never onto. We can see this by viewing this map as multiplication by a matrix of shape \( A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \). Recall that “onto” means that there is always some solution to \( Ax=b \), no matter what value we choose for \( b \). If we set this system up as an augmented matrix and row reduce we must get a bottom row of two zeros followed by whatever the last element of \( b \) becomes. Unless \( b \) is carefully chosen so this last element is zero the system will be inconsistent and for that choice of \( b \) there will be no solution. Geometrically, \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) maps a 2-dim plane into a 3-dim space. At best, the range will be a plane, which cannot be all of \( \mathbb{R}^3 \).

c) The linear map \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is 1-to-1: YES/NO/SOMETIMES

**Answer:** SOMETIMES – Again we view the map as multiplication by the 2x3 matrix we constructed above. Recall that “1-to-1” means that no vector in the image has more than one pre-image. As this map is linear it means that, in particular, the 3-dim zero vector has only one pre-image (the 2-dim zero vector – the trivial solution). This means that, when we row reduce we should have no free variables, or equivalently no non-pivot columns. While the shape of the
matrix suggests that in general there will be no non-pivot columns we can easily build an example which has a non-pivot column, for example \[ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \].

d) Consider the linear map given by multiplication by the matrix \[ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix} \].

i) \( \mathbb{R}^2 \) is the domain/co-domain/range/image

ii) \( \mathbb{R}^3 \) is the domain/co-domain/range/image

**Answer:** The domain is \( \mathbb{R}^3 \) as the matrix equation \( Ax=b \) requires \( x \) to be a 3-vector. Similarly, the co-domain is \( \mathbb{R}^2 \) as the vector \( b \) will be a 2-vector.

e) If \( A \) has a non-pivot column then \( Ax=b \) is consistent: YES/NO/SOMETIMES

**Answer:** SOMETIMES – Whether or not \( Ax=b \) is consistent has more to do with whether or not \( A \) has a non-pivot row than a non-pivot column. Even if \( A \) has a non-pivot row, the consistency of \( Ax=b \) depends on the choice of the vector \( b \). As a simple example, for any matrix \( A \), if the vector \( b \) is the zero vector then the system is consistent.

**Computation Notes:** While these problems are all (obviously) small enough for simple hand computation, similar problems can be done by computer as follows (access to both systems by UMBC students can be found on the home page for this class):

**MAPLE**
We start by loading the linear algebra package:

```maple
with(linalg);
```

For each of the first three problems, once the basic linear system is set up there are several ways to solve it:

```maple
a := matrix([[1,2,-1],[2,1,2],[3,2,1]]);
b := vector([-3,5,3]);
linsolve(a,b);
```

We may also row reduce a matrix with the command:

```maple
rref(a);
```

**MATLAB**
To solve a linear system you use the “\( \backslash \)” operator:

```matlab
A = [1,2,-1;2,1,2;3,2,1]
b = [-3;5;3]
A\b
```

We may also row reduce a matrix with the command:

```matlab
rref(a)
```