Math 100 HW3

1. In each of the following cases explain the difference between the sentences.

(a) • ∀x: ∃y: f(x, y)
   • ∃y: ∀x: f(x, y)

   Soln: In the first sentence we are saying that every x has a corresponding value of y such that f(x, y) is true. This value of y is probably different for different values of x. In the second sentence we are saying that there is some universal y such that f(x, y) is true for all choices of x.

(b) • Everyone has read their own book.
    • Everyone has read a book.
    • There is a book that everyone has read.

   Soln: In the first sentence we are being clear that each person has their own book, strongly implying that no two people share a book. In the third sentence it is clear that it is a single book that everyone shares. The second sentence is worded ambiguously, and it is not clear if the books are shared or not.

2. The popular German game of Skat uses a deck which has 32 cards, divided into four suits, with the cards \{7,8,9,10,J,Q,K,A\} in each suit. A game has three players. At the start of play each player is dealt a hand of 10 cards. In each of the following questions you can write the answer as a product without multiplying it out.

(a) How many different ways are there to order the deck?
   Soln: There are 32! = (32)(31)(30)(29) \ldots (3)(2)(1) = 26313083693369353016721801216000000000 different orderings of the deck.

(b) How many different hands can you be dealt (remember, a hand does not depend on the order you receive the cards)
   Soln: You can hold any one of (32)(31)(30) \ldots (22)(23)/((10)(9)(8) \ldots (3)(2)(1)) = 64512240 different possible 10-card hands from a 32-card deck. (This is more compactly referred to as the binomial coefficient “32 choose 10”, \binom{32}{10} = 32!/(10!(32-10)!!))
(c) How many different hands would have all four aces?
   **Soln:** If a hand has all four aces, there are six other cards in the hand, taken from the remaining \(32 - 4 = 28\) cards in the deck. Thus there are \(\binom{28}{6} = 28!/(6!(28 - 6)!)) = (28)(27)(26)(25)(24)(23)/((6)(5)(4)(3)(2)(1)) = 376740\) such hands.

3. Consider playing a game in which you start the game by each player putting a dollar into a pot, and you flip coins until there are either three heads or three tails. If three heads the first player wins, if three tails, the second player. The game is forced to end when the score is two heads and one tail.
   (a) Sketch a graph showing the outcome, coin flip by coin flip, and giving the likelihood of each outcome.
   **Soln:**

   ![Game Diagram](image)

   (b) What would be a fair division of the pot? Explain and defend your conclusion.
   **Soln:**
   
   - Heads wins when the game reaches the state 3H:1T or the state 3H:2T. The probability of reaching state 3H:1T is \(\frac{1}{2}\) and the probability of reaching state 3H:2T is \(\frac{1}{4}\), so the overall probability of Heads winning is \(\frac{1}{2} + \frac{1}{4} = \frac{3}{4}\).
   - Tails wins when the game reaches the state 2H:3T. The probability of reaching this state is \(\frac{1}{4}\), which is the probability of Tails winning.
   - Thus the pot should be divided in the proportion \(\frac{3}{4}\) to \(\frac{1}{4}\) or 3:1. (In other words, a $100 pot should be divided with $75 to the Heads player and $25 to the Tails player.)