Let $I$ be the identity matrix and let $J$ be the column vector of all ones.

1. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Suppose $B$ is nonsingular and $B^{-1}$ is known.
   
   (a) Show that $M$ is nonsingular if $S = C - DB^{-1}A$ is nonsingular.
   
   (b) Assuming $M$ is nonsingular find $M^{-1}$ in terms of $A, B, B^{-1}, C, D$ and $S^{-1}$.

2. Consider the $n \times n$ matrix $A_{\theta} = \begin{pmatrix} I & \theta J \\ \theta J' & 1 \end{pmatrix}$.
   
   (a) For what values of $\theta$ is $A_{\theta}$ nonsingular ?
   
   (b) Find $A_{\theta}^{-1}$ whenever it exists. Write $A_{\theta}^{-1}$ as $I + B$. Show that $\text{rank}(B) = 2$ and find a rank factorization of $B$, i.e., find $C$ and $D$ such that $B = CD$ and columns of $C$ are linearly independent and the rows of $D$ are linearly independent.

3. Use the result $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \begin{vmatrix} A - BD^{-1}C \end{vmatrix}$ to show that for $A$ nonsingular of order $n \times n$ and $B$ and $C$ or orders $n \times k$ and $k \times n$, respectively,

   $|A - BC| = |A| |I - CA^{-1}B|.$

4. If $A$ is nonnegative definite show that $x'Ax = 0$ iff $y'Ax = 0$. Show also that $x'Ax = 0$ iff $y'Ax = 0$ for $y$.

5. Show that $A = (1 - \rho)I + \rho JJ'$ is positive definite iff $-\frac{1}{n-1} < \rho < 1$ where $n$ is the order of $A$.

6. Let $A$ be $n \times k$, $n \geq k$ and that $\text{rank}(A) = k$.
   
   (a) Show that $A'A$ is nonsingular.

   Let $P = A(A'A)^{-1}A'$. $P$ is called the projection matrix for the column space of $A$, i.e., the vector space generated by all possible linear combinations of the columns of $A$. Show that
   
   i. $\text{rank}(P) = \text{rank}(A)$.
   
   ii. $P' = P$. (symmetric)
   
   iii. $P^2 = P$. (idempotent)
   
   iv. Find the eigenvalues of $P$ and show that $\text{rank}(P) = \text{tr}(P)$.

Result: Any $P$ that is symmetric and idempotent is a Projection matrix, i.e., there exist an $A$ such that $P = A(A'A)^{-1}A'$. For any $x$, $Px$ can be written as a linear combination of the columns of $A$. 1
(b) If $P_A$ is the projection matrix for column space of $A$ then show that $I - P_A$ is also a projection matrix. Write $I - P_A = P_B$. Find the relationship between the column spaces of $A$ and $B$.

Refresh your knowledge on basic probability distributions: Normal, Binomial, Poisson, $\chi^2$, t, F etc. Also recap Conditional Expectation, Conditional Distribution, Law of Large Numbers, Central Limit Theorem, Tchebycheff’s Inequality etc. Read about matrix calculus and other matrix results given in Appendix A. It will also help if you start reading things in Appendix B such as unbiasedness, MLE etc. (You should be familiar with them).