On the P-property of Z and Lyapunov-like transformations on Euclidean Jordan algebras

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On the $P$-property of $\mathbb{Z}$ and Lyapunov-like transformations on Euclidean Jordan algebras.
Outline

- Motivation and a conjecture
- Euclidean Jordan algebras
- $\mathbb{Z}$ and Lyapunov-like transformations
- Validity of the conjecture for Lyapunov-like transformations
- A result for $\mathbb{Z}$-transformations
Recall a result from complementarity problems:

The following are equivalent for $M \in \mathbb{R}^{n \times n}$:

- All principal minors of $M$ are positive.
- $x * Mx \leq 0 \Rightarrow x = 0$.
- LCP$(M, q)$ has a unique solution for all $q \in R^n$.

LCP$(M, q)$: Find $x \in R^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad \text{and} \quad \langle Mx + q, x \rangle = 0.$$
When \( M \) is a z-matrix, i.e., when all off-diagonal entries of \( M \) are non-positive, the above statements are further equivalent to:

- \( \text{LCP}(M, q) \) has a solution for all \( q \).
- There exists a \( d > 0 \) such that \( Md > 0 \).
- \( M \) is positive stable: Real part of any eigenvalue of \( M \) is positive.
$S^n$ - All $n \times n$ real symmetric matrices.

$S^n_+$ - All PSD matrices in $S^n$.

Notation: $X \succeq 0$ if $X \in S^n_+$.

$\langle X, Y \rangle := \text{trace}(XY)$.

$X \circ Y := \frac{XY + YX}{2}$ - Jordan product.

Semidefinite LCP:

$L : S^n \to S^n$ linear, $Q \in S^n$.

SDLCP($L, Q$): Find $X \in S^n$ such that

$$X \succeq 0, \quad L(X) + Q \succeq 0, \quad \text{and} \quad \langle X, L(X) + Q \rangle = 0.$$
For $A \in R^{n \times n}$,

$L_A(X) := AX + XA^T$ - Lyapunov transformation on $S^n$.

$S_A(X) := X - AXA^T$ - Stein transformation on $S^n$.

$L$ denotes either $L_A$ or $S_A$.

The following are equivalent:

- $[XL(X) = L(X)X, \ X \circ L(X) \preceq 0] \Rightarrow X = 0$.
- SDLCP$(L, Q)$ has a solution for all $Q$.
- There exists $D \succ 0$ with $L(D) \succ 0$.
- $L$ is positive stable.
The above result is very similar to the matrix theory result for $\mathbf{Z}$-matrices.

Why is this happening?

Do $L_A$ and $S_A$ have some sort of $\mathbf{Z}$-property?

Can the two results be unified and extended?

Note: Both $\mathcal{R}^n$ and $S^n$ are Euclidean Jordan algebras!
$(V, \langle \cdot, \cdot \rangle, \circ)$ is a Euclidean Jordan algebra if
$V$ is a finite dimensional real inner product space
and the bilinear Jordan product $x \circ y$ satisfies:

- $x \circ y = y \circ x$
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$

$K = \{x^2 : x \in V\}$ is the symmetric cone in $V$.
Notation: $x \geq 0$ if $x \in K$ and $x > 0$ if $x \in \text{int}(K)$. 
Any EJA is a product of the following:

- \( S^n = \text{Herm}(\mathcal{R}^{n \times n}) \) - \( n \times n \) real symmetric matrices.
- \( \text{Herm}(\mathcal{C}^{n \times n}) \) - \( n \times n \) complex Hermitian matrices.
- \( \text{Herm}(\mathcal{Q}^{n \times n}) \) - \( n \times n \) quaternion Hermitian matrices.
- \( \text{Herm}(\mathcal{O}^{3 \times 3}) \) - \( 3 \times 3 \) octonion Hermitian matrices.
- \( \mathcal{L}^n \) - Jordan spin algebra.

For \( a \in V \), \( L_a(x) := a \circ x \).

\( a \) and \( b \) operator commute if \( L_aL_b = L_bL_a \).
Let $L$ be linear on $V$ and $q \in V$.

$LCP(L, K, q) : x \geq 0, \ L(x) + q \geq 0, \ \langle L(x) + q, x \rangle = 0$.

- **GUS**-property: Unique solution in all $LCP(L, K, q)$.
- **P**-property:
  \[ [x \text{ and } L(x) \text{ operator commute, } x \circ L(x) \leq 0] \Rightarrow x = 0. \]
- **Q**-property: For all $q \in V$, $LCP(L, K, q)$ has a solution.
- **S**-property: There exists $d > 0$ such that $L(d) > 0$.

Gowda, Sznajder, Tao (2004):

\[ \text{GUS} \Rightarrow \text{P} \Rightarrow \text{Q} \Rightarrow \text{S}. \]
- **Z-property:** \([x, y \in K, x \perp y] \Rightarrow \langle L(x), y \rangle \leq 0.\)
- **Lyapunov-like:** \([x, y \in K, x \perp y] \Rightarrow \langle L(x), y \rangle = 0.\)

**Example:** \(L_A\) is Lyapunov-like and \(S_A\) has Z-property.
\((L_A(X) = AX + XA^T\) and \(S_A(X) = X - AXA^T\) on \(S^n\).)

**Schneider-Vidyasagar (1970)**
Z-property is equivalent to:

- \(exp(-tL)(K) \subseteq K\) for all \(t \geq 0.\)
- \(\dot{x} + L(x) = 0, x(0) \in K \Rightarrow x(t) \in K\) for all \(t \geq 0.\)
Stern (1981), Gowda-Tao (2009): For a Z-transformation, the following are equivalent:

- **s-property**
- Positive stable property
- $L^{-1}(K) \subseteq K$.
- **Q-property**

**Conjecture:** For a Z-transformation, $P=Q$

Conjecture holds for matrices on $\mathcal{R}^n$, $L_A$ and $S_A$ on $S^n$. 
Will show:

Conjecture holds for all Lyapunov-like transformations and those $\mathbf{Z}$-transformations with $L(e) > 0$, where $e$ is the unit element in $V$. 
New Results

A characterization of Lyapunov-like transformations:
The following are equivalent:

- $L$ is Lyapunov-like.
- $e^{tL}(K) = K$ for all $t \in \mathcal{R}$.
- $L$ belongs to the Lie algebra of $\text{Aut}(K)$.
- $L = L_a + D$, where $L_a(x) = a \circ x$ and $D$ is a derivation.

Derivation:

$$D(x \circ y) = D(x) \circ y + x \circ D(y) \text{ for all } x, y \in V.$$
Theorem A

For a Lyapunov-like transformation, $P=Q$.

A sketch of the Proof:
Assume $L$ is Lyapunov-like and positive stable.
Suppose $x \neq 0$ operator commutes with $L(x)$ and $x \circ L(x) \leq 0$.
Write spectral decompositions
$x = \sum x_i e_i$ and $L(x) = \sum y_i e_i$
with $x_i y_i \leq 0$ for all $i$ and $x_i \neq 0$ for $i = 1, 2, \ldots k$. 
Let $c := e_1 + e_2 + \cdots + e_k$ and $W := \{x : x \circ c = x\}$.

Then $L(W) \subseteq W$ and so restriction $L'$ of $L$ to $W$ is also positive stable.

Thus $L'$ has positive trace.

But the Lyapunov-like property together with $x_i y_i \leq 0$ for all $i$ implies that trace of $L'$ is non-positive.

This is a contradiction.
Theorem B

Let $L$ be a $\mathbb{Z}$-transformation with $L(e) > 0$. Then $\textbf{P} = \textbf{Q}$.

Sketch of the proof:

Suppose $x \neq 0$ operator commutes with $L(x)$ and $x \circ L(x) \leq 0$.

Write $x = \sum x_i e_i$ and $L(x) = \sum y_i e_i$

with $x_i y_i \leq 0$ for all $i$. 
Define $A = [a_{ij}]$, where $a_{ij} := \langle L(e_i), e_j \rangle$.

Then $A$ is a $\mathbf{Z}$-matrix, $Au > 0$, where $u$ is the vector of ones and $p \ast A^T p \leq 0$ in $\mathbb{R}^n$, where $p$ is the vector with components $x_i$.

By matrix theory results, $A$ is a $\mathbf{P}$-matrix.

Hence $A^T$ is a $\mathbf{P}$-matrix and $p \ast A^T p \leq 0 \Rightarrow p = 0$,

This implies that $x = 0$, leading to a contradiction.
Open problems

- **Conjecture**: For any $Z$-transformation, $P = Q$.
- Characterize the **GUS**-property for $Z$-transformations.
- When is $S_A$ **GUS**?